

CLASS XII (2019-20)
MATHEMATICS (041)
SAMPLE PAPER-3

Time : 3 Hours**Maximum Marks : 80****General Instructions :**

- (i) All questions are compulsory.
- (ii) The questions paper consists of 36 questions divided into 4 sections A, B, C and D.
- (iii) Section A comprises of 20 questions of 1 mark each. Section B comprises of 6 questions of 2 marks each. Section C comprises of 6 questions of 4 marks each. Section D comprises of 4 questions of 6 marks each.
- (iv) There is no overall choice. However, an internal choice has been provided in three questions of 1 mark each, two questions of 6 marks each. You have to attempt only one of the alternatives in all such questions.
- (v) Use of calculators is not permitted.

Section-A

DIRECTION : (Q 1-Q 10) are multiple choice type questions. Select the correct option.

1. If $f: R \rightarrow R$ such that $f(x) = 3x - 4$ then which of the following is $f^{-1}(x)$? [1]

- (a) $\frac{x+4}{3}$ (b) $\frac{1}{3}x - 4$
 (c) $3x - 4$ (d) $3x + 5$

Ans : (a) $\frac{x+4}{3}$

Let, $f(x) = y = 3x - 4$
 $\frac{y+4}{3} = x$

Replacing $x \rightarrow y$ and $y \rightarrow x$

we get, $f^{-1}(x) = \frac{x+4}{3}$

2. If $2 \begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, then- [1]

- (a) $(x = -2, y = 8)$ (b) $(x = 2, y = -8)$
 (c) $(x = 3, y = -6)$ (d) $(x = -3, y = 6)$

Ans : (b) $(x = 2, y = -8)$

We have $2 \begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$
 $\begin{bmatrix} 6+1 & 8+y \\ 10+0 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$
 $\begin{bmatrix} 7 & 8+y \\ 10 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$

$8 + y = 0$ and $2x + 1 = 5$
 $y = -8$ and $x = 2$

3. The matrix $\begin{bmatrix} 3 & 5 \\ 2 & k \end{bmatrix}$ has no inverse if the value of k is [1]

- (a) 0 (b) 5
 (c) $\frac{10}{3}$ (d) $\frac{4}{9}$

Ans : (c) $\frac{10}{3}$

$$A = \begin{bmatrix} 3 & 5 \\ 2 & k \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

A^{-1} will not exist if $|A| = 0$.

Therefore, $3k - 10 = 0$

$$k = \frac{10}{3}$$

Hence, inverse will not exist if $k = \frac{10}{3}$.

4. $\frac{d}{dx} [\log(\sec x + \tan x)] =$ [1]

- (a) $\frac{1}{\sec x + \tan x}$ (b) $\sec x$
 (c) $\tan x$ (d) $\sec x + \tan x$

Ans : (b) $\sec x$

Let, $y = \log(\sec x + \tan x)$
 $t = \sec x + \tan x$

Now, $y = \log t$
 Differentiating both sides, with respect to t , we get

$$\frac{dy}{dt} = \frac{1}{t}$$

Now, $t = \sec x + \tan x$
 Differentiating both sides with respect to x

, we get

$$\begin{aligned} \frac{dt}{dx} &= \sec x \cdot \tan x + \sec^2 x \\ &= \sec x(\tan x + \sec x) \end{aligned}$$

Now, $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$\begin{aligned} &= \frac{1}{t} \times \sec x(\tan x + \sec x) \\ &= \frac{1}{(\sec x + \tan x)} \times \sec x(\tan x + \sec x) \\ &= \sec x \end{aligned}$$

5. The slope of the tangent to the curve, $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $(2, -1)$ is- [1]

- (a) $\frac{12}{7}$ (b) $-\frac{6}{7}$
 (c) $\frac{6}{7}$ (d) $-\frac{12}{7}$

Ans : (c) $\frac{6}{7}$

We have,

$$\begin{aligned} t^2 + 3t - 8 = 2 \text{ and } 2t^2 - 2t - 5 = -1 \\ t^2 + 3t - 10 = 0 \text{ and } 2t^2 - 2t - 4 = 0 \\ (t + 5)(t - 2) = 0 \text{ and } t^2 - t - 2 = 0 \\ t = -5, 2 \end{aligned}$$

and $(t - 2)(t + 1) = 0$
 $t = -5, 2$ and $t = 2, -1$
 $t = 2$

Now, $\frac{dy}{dt} = 4t^2 - 2$ and $\frac{dx}{dt} = 2t + 3$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right) / \left(\frac{dx}{dt}\right) = \frac{4t^2 - 2}{2t + 3}$$

Slope of the tangent

$$\begin{aligned} \left(\frac{dy}{dx}\right)_{(2,-1)} &= \frac{4t - 2}{2t + 3} \\ &= \frac{4(2) - 2}{2 \times 2 + 3} \quad [t = 2] \\ &= \frac{8 - 2}{4 + 3} = \frac{6}{7} \end{aligned}$$

6. $\int_0^1 \frac{(\tan^{-1} x)^2}{1 + x^2} dx =$ [1]

- (a) 1 (b) $\frac{\pi^3}{64}$
 (c) $\frac{\pi^2}{192}$ (d) None of these

Ans : (d) None of these

Let, $I = \int_0^1 \frac{(\tan^{-1} x)^2}{1 + x^2} dx$

Put, $\tan^{-1} x = t$

and $\frac{1}{1 + x^2} dx = dt$

When, $x = 0, t = 0$

and $x = 1, t = \frac{\pi}{4}$

Thus, $I = \int_0^{\frac{\pi}{4}} t^2 dt = \frac{t^3}{3} \Big|_0^{\pi/4}$

$$= \frac{1}{3} \left(\frac{\pi}{4}\right)^3 - \frac{0}{3} = \frac{\pi^3}{192}$$

7. Solution of the differential equation $ydx - xdy = xydx$ is [1]

- (a) $\frac{y^2}{2} - \frac{x^2}{2} = xy + c$ (b) $x = kye^x$
 (c) $x = kye^y$ (d) None of these

Ans : (b) $x = kye^x$

$$ydx - xdy = xydx$$

Dividing both sides by y^2

$$\frac{ydx - xdy}{y^2} = \frac{xy}{y^2} dx \quad \dots(1)$$

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

Putting value in eq. (1),

$$d\left(\frac{x}{y}\right) = \frac{x}{y} dx \quad \dots(2)$$

Putting $\frac{x}{y} = t$ in eq.(2),

$$dt = t dx$$

$$\frac{1}{t} dt = dx$$

Integrate in both sides

$$\int \frac{1}{t} dt = \int dx$$

$$\ln t = x + c$$

$$t = e^{x+c} = e^x \cdot e^c = ke^x$$

$$[e^c = k = \text{constant}]$$

Substituting $t = \frac{x}{y}$,

$$\frac{x}{y} = ke^x$$

$$x = kye^x$$

8. If $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$, then the value of $(\vec{a} + 3\vec{b}) \cdot (2\vec{a} - \vec{b})$ is- [1]

- (a) 15 (b) 18
 (c) -18 (d) -15

Ans : (d) -15

We have,

$$(\vec{a} + 3\vec{b}) \cdot (2\vec{a} - \vec{b}) = 2\vec{a} \cdot \vec{a} + 6\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{b} - 3\vec{b} \cdot \vec{b}$$

$$= 2|\vec{a}|^2 + 5\vec{a} \cdot \vec{b} - 3|\vec{b}|^2$$

Here $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$
 $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$

$$= 2(\sqrt{1^2 + 1^2 + 2^2})^2 + 5(1 \times 3 + 1 \times 2 + 2 \times (-1)) - 3(\sqrt{3^2 + 2^2 + (-1)^2})^2$$

$$= 2 \times 6 + 5 \times 3 - 3 \times 14$$

$$= 12 + 15 - 42 = 27 - 42 = -15$$

9. The direction ratios of a straight line are 1,3,5. Its direction cosines are [1]

- (a) $\frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}$ (b) $\frac{1}{9}, \frac{1}{3}, \frac{5}{9}$
 (c) $\frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{1}{\sqrt{35}}$ (d) $\frac{5}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{1}{\sqrt{35}}$

Ans : (a) $\frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}$

If l, m, n are direction cosine of line and a, b, c are direction ratios then

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Given direction ratios are

$$a = 1, b = 3, c = 5$$

Then, $a^2 + b^2 + c^2 = 1 + 9 + 25 = 35$

Therefore, $l = \frac{1}{\sqrt{35}}, m = \frac{3}{\sqrt{35}}$ and

$$n = \frac{5}{\sqrt{35}}$$

10. If $P(A) = \frac{3}{8}, P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$ then $P(A' \cap B')$ [1]

- (a) $\frac{13}{8}$ (b) $\frac{13}{4}$
 (c) $\frac{13}{24}$ (d) $\frac{13}{9}$

Ans : (c) $\frac{13}{24}$

We have $P(A) = \frac{3}{8}$

$$P(B) = \frac{1}{3}$$

$$P(A \cap B) = \frac{1}{4}$$

$$P(A' \cap B') = P(\overline{A \cup B})$$

We know that,

$$= P(A) + P(B) - P(A \cap B)$$

$$= \frac{3}{8} + \frac{1}{3} - \frac{1}{4} = \frac{11}{24}$$

$$P(\overline{A \cup B}) + P(A \cap B) = 1$$

$$P(\overline{A \cup B}) = 1 - P(A \cap B)$$

$$= 1 - \frac{11}{24} = \frac{24 - 11}{24}$$

$$= \frac{13}{24}$$

Q. 11-15 (Fill in the blanks)

11. Let \vec{a} and \vec{b} be two given vectors such that $|\vec{a}| = 2, |\vec{b}| = 1$ and $\vec{a} \cdot \vec{b} = 1$. The angle between \vec{a} and \vec{b} [1]

Ans :

Let θ be the angle between \vec{a} and \vec{b} . Then,

$$\vec{a} \cdot \vec{b} = 1 \Rightarrow |\vec{a}| |\vec{b}| \cos \theta = 1$$

$$\Rightarrow (2 \times 1) \cos \theta = 1 \left[\because |\vec{a}| = 2 \text{ and } |\vec{b}| = 1 \right]$$

$$(1/2)$$

$$\Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Hence the angel between \vec{a} and \vec{b} is $\frac{\pi}{3}$. (1/2)

12. If I_3 is the identity matrix of order 3 then the value of $(3I_3)$ will be [1]

Ans :

Since the matrix I_3 is of order 3×3 , therefore,

$$|3I_3| = 3^3 |I_3| \quad (1/2)$$

$$\Rightarrow |3I_3| = 27 |I_3|$$

$$= 27(1) = 27 \quad (1/2)$$

13. The principal value of $\operatorname{cosec}^{-1}(2)$ will be [1]

Ans :

Let $\operatorname{cosec}^{-1}(2) = \theta$

$$\Rightarrow \operatorname{cosec} \theta = 2$$

i.e. $\operatorname{cosec} \theta = \operatorname{cosec}\left(\frac{\pi}{6}\right) \quad (1/2)$

So, principle value of $\operatorname{cosec}^{-1}(2)$ is $\frac{\pi}{6}$ as principal value of y

$$= \operatorname{cosec}^{-1} x \text{ is } \left[\frac{-\pi}{2}, \frac{\pi}{2} \right] - \{0\}. (1/2)$$

14. If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \forall x_1, x_2 \in A$, then the function $f: A \rightarrow B$ is [1]

- (a) one-one (b) constant

(c) onto (d) many one

Ans : (a) one-one

If $f(x_1) = f(x_2)$
 $x_1 = x_2 \forall x_1, x_2 \in A$

Since there exist a unique value of $f(x)$ in set B Therefore $f: A \rightarrow B$ is one-one.

or

If function $f: N \rightarrow N$ be defined by $f(x) = 4x + 3$ then $f^{-1}(x) = \dots\dots\dots$

(a) $4x - 3$ (b) $\frac{4x - 3}{2}$

(c) $\frac{x + 3}{2}$ (d) $\frac{x - 3}{4}$

Ans : (d) $\frac{x - 3}{4}$

$$f: N \rightarrow N$$

$$f(x) = 4x + 3$$

Let, $y = 4x + 3$

Then, $y - 3 = 4x$

$$x = \frac{y - 3}{4}$$

$$y \rightarrow x \text{ and } x \rightarrow y$$

We get $f^{-1}(x) = \frac{x - 3}{4}$

15. The order of the differential equation $\left(\frac{dy}{dx}\right)^2 + y = x$ is [1]

(a) 0 (b) 1
 (c) 2 (d) 3

Ans : (b) 1

Order of differential equation is the order of highest differential coefficient occurring in it.

Therefore, order of $\left(\frac{dy}{dx}\right)^2 + y = x$ is 1

or

The differential equation of family of lines passing through the origin is

(a) $x\frac{dy}{dx} = y$ (b) $y\frac{dy}{dx} = x$

(c) $\frac{dy}{dx} = y$ (d) $\frac{dy}{dx} = x$

Ans : (a) $x\frac{dy}{dx} = y$

Equation of any line passing through origin is given by,

$$y = mx \dots(i)$$

on differentiating Eq. (i) w.r.t. x , we get

$$\frac{dy}{dx} = m \dots(ii)$$

From Eq. (i) and (ii), we have

$$y = x\frac{dy}{dx}$$

which is required differential equation.

16. If A is a matrix of order 2×3 and B is a matrix of order 3×5 , then what is the order of matrix $(AB)'$ or $(AB)^T$? [1]

Ans :

Given, A is a matrix of order 2×3 and B is a matrix of order 3×5 , therefore the product AB is a matrix of order 2×5 . Then, the order of matrix $(AB)'$ or $(AB)^T$ is 5×2 .

17. Find the value of λ , so that the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + \lambda\hat{j} + 3\hat{k}$ are perpendicular to each other. [1]

Ans :

Given, $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$

and $\vec{b} = \hat{i} + \lambda\hat{j} + 3\hat{k}$

Since, vectors \vec{a} and \vec{b} are perpendicular to each other.

$$\therefore \vec{a} \cdot \vec{b} = 0$$

$$(3\hat{i} + 2\hat{j} + 9\hat{k}) \cdot (\hat{i} + \lambda\hat{j} + 3\hat{k}) = 0$$

$$3 + 2\lambda + 27 = 0$$

$$2\lambda = -30$$

$$\Rightarrow \lambda = -\frac{30}{2}$$

$$= -15$$

Hence, the required value of λ is -15 .

18. Let $f: R \rightarrow R$, $f(x) = (x^2 - 3x + 2)$. Find $f \circ f(x)$. [1]

Ans :

$$\begin{aligned} f \circ f(x) &= f\{f(x)\} = f\{x^2 - 3x + 2\} \\ &= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2 \\ &= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 \\ &\quad - 3x^2 + 9x - 6 + 2 \\ &= x^4 - 6x^3 + 10x^2 - 3x \end{aligned}$$

19. Prove that the function f given by $f(x) = \log \cos x$ is strictly decreasing. [1]

Ans :

Given, $f(x) = \log \cos x$

On differentiating both sides w.r.t. x , we get

$$f'(x) = \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x)$$

$$= \frac{1}{\cos x} \cdot (-\sin x)$$

$$= -\tan x$$

We know that, for $x \in (0, \frac{\pi}{2}), \tan x > 0$

$$\therefore f'(x) = -\tan x < 0$$

Hence, $f(x)$ is strictly decreasing.

Hence proved

20. Maximise $Z = 3x + 4y$, subject to the constraints $x + y \leq 1, x \geq 0, y \geq 0$. [1]

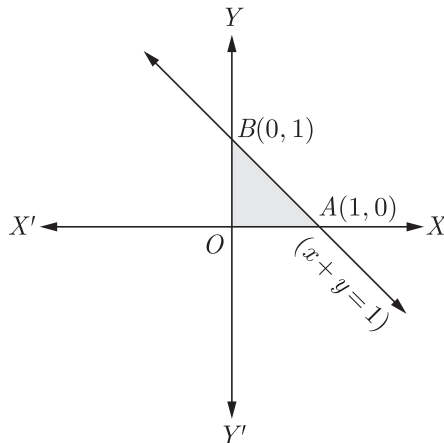
Ans :

Maximize $Z = 3x + 4y$

Subject to the constraints

$$x + y \leq 1, x \geq 0, y \geq 0$$

The shaded region shown in the figure as OAB is bounded and the coordinates of corner points O, A and B are $(0,0), (1,0)$ and $(0,1)$, respectively.



Corner points	Corresponding value of Z
$(0, 0)$	0
$(1, 0)$	3
$(0, 1)$	4 ← Maximum

Hence, the maximum value of Z is 4 at $(0,1)$.

Section B

21. Solve for $x \cos(2\sin^{-1}x) = \frac{1}{9}, x > 0$ [2]

Ans :

We have, $\cos(2\sin^{-1}x) = \frac{1}{9}$

We know that,

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos(2\sin^{-1}x) = 1 - 2\sin^2(\sin^{-1}x)$$

$$= \frac{1}{9} \tag{1}$$

$$1 - 2x^2 = \frac{1}{9}$$

$$2x^2 = 1 - \frac{1}{9}$$

$$2x^2 = \frac{8}{9}$$

$$x^2 = \frac{4}{9} \Rightarrow x = \frac{2}{3} \tag{1}$$

or

Evaluate $\cos\left[\sin^{-1}\frac{1}{4} + \sec^{-1}\frac{4}{3}\right]$

Ans :

We have, $\cos\left[\sin^{-1}\frac{1}{4} + \sec^{-1}\frac{4}{3}\right]$

$$= \cos\left[\sin^{-1}\frac{1}{4} + \cos^{-1}\frac{3}{4}\right] \tag{1/2}$$

$$= \cos\left(\sin^{-1}\frac{1}{4}\right)\cos\left(\cos^{-1}\frac{3}{4}\right) - \sin\left(\sin^{-1}\frac{1}{4}\right)$$

$$\sin\left(\cos^{-1}\frac{3}{4}\right)$$

$$= \frac{3}{4}\sqrt{1 - \left(\frac{1}{4}\right)^2} - \frac{1}{4}\sqrt{1 - \left(\frac{3}{4}\right)^2} \tag{1/2}$$

$$= \frac{3}{4} \times \frac{\sqrt{15}}{4} - \frac{1}{4} \times \frac{\sqrt{7}}{4}$$

$$= \frac{3\sqrt{15} - \sqrt{7}}{16} \tag{1}$$

22. Find the derivative of $\log \sin x$ w.r.t. x . [2]

Ans :

Let $y = \log \sin x$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) \text{ (by chain rule)}$$

$$\tag{1/2}$$

$$= \frac{1}{\sin x} \cdot \cos x \tag{1/2}$$

$$\Rightarrow \frac{dy}{dx} = \cot x \tag{1}$$

23. Evaluate $\int (3 \operatorname{cosec}^2 x - 5x + \sin x) dx$. [2]

Ans :

Let

$$I = \int (3 \operatorname{cosec}^2 x - 5x + \sin x) dx$$

$$= 3 \int \operatorname{cosec}^2 x dx - 5 \int x dx + \int \sin x dx \tag{1/2}$$

$$= 3(-\cot x) - 5\frac{x^2}{2} - \cos x + C \tag{1}$$

$$= -3\cot x - \frac{5x^2}{2} - \cos x + C \tag{1/2}$$

24. If the function $f(x) = \frac{1}{x+2}$, find the points of discontinuity of the composite function $y = f(f(x))$. [2]

Ans :

Given, $f(x) = \frac{1}{x+2}$

Clearly, $f(x)$ is not continuous at $x = -2$
(1/2)

[∵ rational functions are continuous for all real numbers except at those points, where the denominator is zero]

Now, for $x \neq -2$

$$\begin{aligned} f(f(x)) &= \frac{1}{f(x)+2} = \frac{1}{\frac{1}{(x+2)}+2} \\ &= \frac{(x+2)}{1+2(x+2)} \\ &= \frac{x+2}{2x+5} \end{aligned}$$

which is discontinuous at $x = -\frac{5}{2}$ (1)

Hence, the points of discontinuity are

$$x = -2 \text{ and } x = -\frac{5}{2} \quad (1/2)$$

or

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ and $x \neq y$, prove that $\frac{dy}{dx} = -\frac{1}{(x+1)^2}$.

Ans :

Given $x\sqrt{1+y} + y\sqrt{1+x} = 0$
 $x\sqrt{1+y} = -y\sqrt{1+x}$
 $x^2(1+y) = y^2(1+x)$
 (squaring both sides)

$$x^2 - y^2 = y^2x - x^2y \quad (1/2)$$

$$(x+y)(x-y) = -xy(x-y)$$

$$\Rightarrow x+y = -xy$$

$$x = -y - xy$$

$$\Rightarrow y(1+x) = -x$$

$$y = -\frac{x}{1+x} \quad (1/2)$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -\left[\frac{(1+x) \cdot 1 - x(0+1)}{(1+x)^2} \right] \\ &= -\frac{1}{(1+x)^2} \text{ Hence proved} \end{aligned} \quad (1)$$

25. Without expanding, show that [2]

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0$$

Ans :

Applying $C_1 \rightarrow C_1 - C_2 - C_3$, we have

$$\Delta \begin{vmatrix} \operatorname{cosec}^2\theta - \cot^2\theta - 1 & \cot^2\theta & 1 \\ \cot^2\theta - \operatorname{cosec}^2\theta + 1 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} \quad (1)$$

$$= \begin{vmatrix} 0 & \cot^2\theta & 1 \\ 0 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = 0 \quad (1)$$

26. Show that $\Delta = \begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix} = (x-p)(x^2 + px - 2q^2)$ [2]

Ans :

Applying $C_1 \rightarrow C_1 - C_2$, we have

$$(1) \quad \Delta = \begin{vmatrix} x-p & p & q \\ p-x & x & q \\ 0 & q & x \end{vmatrix} = (x-p) \begin{vmatrix} 1 & p & q \\ -1 & x & q \\ 0 & q & x \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2$

$$= (x-p) \begin{vmatrix} 0 & p+x & 2q \\ -1 & x & q \\ 0 & q & x \end{vmatrix} \quad (1/2)$$

Expanding along C_1 , we have

$$\begin{aligned} \Delta &= (x-p)(px + x^2 - 2q^2) \\ &= (x-p)(x^2 + px - 2q^2) \end{aligned} \quad (1/2)$$

Section C

27. Let $f: R \rightarrow R$ defined by $f(x) = \frac{2x-1}{3}$, $x \in R$, where x is the number of students in a class and $f(x)$ is money collected by the class for girl child welfare, Show that f is invertible. [4]

Ans :

(1) Let $x_1, x_2 \in R$ such that

$$f(x_1) = f(x_2)$$

$$\Rightarrow \frac{2x_1-1}{3} = \frac{2x_2-1}{3}$$

$$\Rightarrow 2x_1 - 1 = 2x_2 - 1 \quad (1/2)$$

$$\Rightarrow x_1 = x_2 \quad (1/2)$$

Thus, f is one-one

Now, consider $y \in R$, such that

$$\Rightarrow y = f(x)$$

$$\Rightarrow y = \frac{2x-1}{3} \quad (1/2)$$

$$\Rightarrow 3y = 2x - 1$$

$$\Rightarrow x = \frac{3y+1}{2} \in R \quad (1/2)$$

Also, $f(x) = f\left(\frac{3y+1}{2}\right)$

$$= \frac{2\left(\frac{3y+1}{2}\right) - 1}{3}$$

$$= \frac{3y+1-1}{3} = y \quad (1/2)$$

∴ For every $y \in R$, there exists $x \in R$ such that $f(x) = y$ (1/2)

⇒ Every element in co-domain has its pre-image in domain.

Thus, f is onto.

Hence, f is both one-one and onto

⇒ f is invertible. (1)

28. Solve the differential equation $\frac{dy}{dx} + \frac{y}{x} = x^2$. [4]

Ans :

The equation is of the type $\frac{dy}{dx} + Py = Q$, which is a linear differential equation. (1)

Now, I.F. = $e^{\int \frac{1}{x} dx} = e^{\log x} = x$. (1)

Therefore, solution of the given differential equation is

$$y \cdot x = \int x^2 dx, \quad (2)$$

$$yx = \frac{x^4}{4} + c$$

Hence, $y = \frac{x^3}{4} + \frac{c}{x}$
or

Solve $x^2 \frac{dy}{dx} - xy = 1 + \cos\left(\frac{y}{x}\right)$, $x \neq 0$ and

$$x = 1, y = \frac{\pi}{2}.$$

Ans :

Given equation can be written as

$$x^2 \frac{dy}{dx} - xy = 2 \cos^2\left(\frac{y}{2x}\right), x \neq 0 \quad (1/2)$$

$$\Rightarrow \frac{x^2 \frac{dy}{dx} - xy}{2 \cos^2\left(\frac{y}{2x}\right)} = 1$$

$$\Rightarrow \frac{\sec^2\left(\frac{y}{2x}\right)}{2} \left[x^2 \frac{dy}{dx} - xy \right] = 1 \quad (1)$$

Dividing both sides by x^3 , we get

$$\frac{\sec^2\left(\frac{y}{2x}\right)}{2} \left[\frac{x \frac{dy}{dx} - y}{x^2} \right] = \frac{1}{x^3}$$

$$\Rightarrow \frac{d}{dx} \left[\tan\left(\frac{y}{2x}\right) \right] = \frac{1}{x^3} \quad (1)$$

Integrating both sides, we get

$$\tan\left(\frac{y}{2x}\right) = \frac{-1}{2x^2} + k \quad (1)$$

Substituting $x = 1, y = \frac{\pi}{2}$, we get

$k = \frac{3}{2}$, therefore, $\tan\left(\frac{y}{2x}\right) = -\frac{1}{2x^2} + \frac{3}{2}$ is the required solution. (1/2)

29. Find the values of x which satisfy the equation: [4]

$$\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x.$$

Ans :

From the given equation, we have

$$\sin(\sin^{-1}x + \sin^{-1}(1-x)) = \sin(\cos^{-1}x)$$

$$\Rightarrow \sin(\sin^{-1}x)\cos(\sin^{-1}(1-x)) + \cos(\sin^{-1}x)$$

$$\sin(\sin^{-1}(1-x)) = \sin(\cos^{-1}x) \quad 1\frac{1}{2}$$

$$\Rightarrow x\sqrt{1-(1-x)^2} + (1-x)\sqrt{1-x^2} = \sqrt{1-x^2}$$

$$\Rightarrow x\sqrt{2x-x^2} + \sqrt{1-x^2}(1-x-1) = 0 \quad (1\frac{1}{2})$$

$$\Rightarrow x(\sqrt{2x-x^2} - \sqrt{1-x^2}) = 0$$

$$\Rightarrow x = 0 \text{ or } 2x-x^2 = 1-x^2 \quad (1/2)$$

$$\Rightarrow x = 0 \text{ or } x = \frac{1}{2} \quad (1/2)$$

30. Find the equation of the plane passing through the points $(2,1,-1)$ and $(-1,3,4)$ and perpendicular to the plane $x-2y+4z=10$. [4]

Ans :

The equation of any plane through $(2,1,-1)$ is (1)

$$a(x-2) + b(y-1) + c(z+1) = 0 \quad \dots(i)$$

If it passes through $(-1,3,4)$, then

$$a(-1-2) + b(3-1) + c(4+1) = 0$$

$$\Rightarrow -3a + 2b + 5c = 0 \quad \dots(ii)$$

If plane (i) is perpendicular to the plane

$$x-2y+4z=10, \text{ then}$$

$$a-2b+4c=0 \quad (\because a_1a_2+b_1b_2+c_1c_2=0)$$

$$\dots(iii) \quad (1)$$

On solving eqs. (ii) and (iii) by the method of cross-multiplication, we get

$$\frac{a}{8+10} = \frac{b}{5+12} = \frac{c}{6-2}$$

$$\Rightarrow \frac{a}{18} = \frac{b}{17} = \frac{c}{4} = \lambda \text{ (say)}$$

$$\Rightarrow a = 18\lambda, b = 17\lambda, c = 4\lambda \quad (1)$$

On putting $a = 18\lambda, b = 17\lambda$ and $c = 4\lambda$ in eq. (i), we get

$$18\lambda(x-2) + 17\lambda(y-1) + 4\lambda(z+1) = 0$$

$$\Rightarrow 18x + 17y + 4z = 49 \quad (1)$$

This is the required equation of the plane.

- 31.** Find the unit vector in the direction of the sum of the vectors $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + 3\hat{k}$. [4]

Ans :

Let \vec{c} denote the sum of \vec{a} and \vec{b} . We have

$$\begin{aligned} \vec{c} &= (2\hat{i} - \hat{j} + 2\hat{k}) + (-\hat{i} + \hat{j} + 3\hat{k}) \\ &= \hat{i} + 5\hat{k} \end{aligned} \quad (1)$$

Now, $|\vec{c}| = \sqrt{1^2 + 5^2} = \sqrt{26} \quad (1)$

Thus, the required unit vector is

$$\begin{aligned} \hat{c} &= \frac{\vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{26}}(\hat{i} + 5\hat{k}) \quad (1) \\ &= \frac{1}{\sqrt{26}}\hat{i} + \frac{5}{\sqrt{26}}\hat{k} \end{aligned} \quad (1)$$

or

If \vec{a} , \vec{b} and \vec{c} determine the vertices of a triangle, show that $\frac{1}{2}[\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}]$

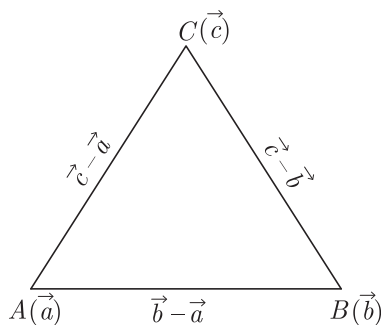
gives the vector area of the triangle. Hence, deduce the condition that the three points \vec{a} , \vec{b} and \vec{c} are collinear. Also, find the unit vector normal to the plane of the triangle.

Ans :

Let \vec{a} , \vec{b} and \vec{c} be the vertices of a ΔABC .

$$\vec{AB} = \vec{b} - \vec{a}$$

and $\vec{AC} = \vec{c} - \vec{a}$



Now, area of $\Delta ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}|$

$$= \frac{1}{2}|(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})| \quad (1)$$

$$= \frac{1}{2}|\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}|$$

$$= \frac{1}{2}|\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a} + \vec{0}|$$

$$\begin{aligned} [\because \vec{b} \times \vec{a} = -\vec{a} \times \vec{b}, \vec{a} \times \vec{c} = -\vec{c} \times \vec{a} \\ \text{and } \vec{a} \times \vec{a} = \vec{0}] \\ = \frac{1}{2}|\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}| \end{aligned} \quad (1)$$

Hence, $\frac{1}{2}[\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}]$ gives the vector area of the triangle.

If these three points are collinear, the area of ΔABC should be equal zero.

$$\frac{1}{2}[\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}] = 0$$

$$\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b} = 0 \quad (1)$$

This is the required condition for collinearity of three points \vec{a} , \vec{b} and \vec{c} .

Let \hat{n} be the unit vector normal to the plane of the ΔABC .

Then, $\hat{n} = \frac{\vec{AB} \times \vec{AC}}{|\vec{AB} \times \vec{AC}|}$

$$= \frac{(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})}{|(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})|}$$

$$= \frac{\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}}{|\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}|}$$

$$= \frac{\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}}{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|} \quad (1)$$

- 32.** Find the vector equation of a line passing through a point with position vector $2\hat{i} - \hat{j} + \hat{k}$, and parallel to the line joining the points $-\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$. Also, find the Cartesian equivalent of this equation. [4]

Ans :

Let A, B, C be the points with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $-\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$ respectively.

We have to find the equation of a line passing through the point A and parallel to \vec{BC} (1)

We have,

$$\begin{aligned} \vec{BC} &= \text{Position vector of } C \\ &\quad - \text{Position vector of } B \end{aligned}$$

$$\begin{aligned} \Rightarrow \vec{BC} &= (\hat{i} + 2\hat{j} + 2\hat{k}) - (-\hat{i} + 4\hat{j} + \hat{k}) \\ &= 2\hat{i} - 2\hat{j} + \hat{k} \end{aligned}$$

We know that the equation of a line passing through a point \vec{a} and parallel to \vec{b} is

$$\vec{r} = \vec{a} + \lambda\vec{b} \quad (1/2)$$

Here, $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$.

So, the equation of the required line is

$$\vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$$

...(i) (1/2)

Reduction to cartesian form: Putting $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ in (1), we obtain

$$\begin{aligned} x\vec{i} + y\vec{j} + z\vec{k} &= (2 + 2\lambda)\vec{i} + (-1 - 2\lambda)\vec{j} + (1 + \lambda)\vec{k} \\ \Rightarrow x &= 2 + 2\lambda, y = -1 - 2\lambda, z = 1 + \lambda \end{aligned} \tag{1}$$

$$\Rightarrow \frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1},$$

which is the Cartesian equivalent of equation (i) (1)

Section D

33. Show that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix. [6]

Ans :

Let A be a square matrix. Then,

$$\begin{aligned} A &= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) \\ &= P + Q \text{ (say),} \end{aligned} \tag{1\frac{1}{2}}$$

where $P = \frac{1}{2}(A + A^T)$ and $Q = \frac{1}{2}(A - A^T)$

$$\begin{aligned} \text{Now, } P^T &= \left(\frac{1}{2}(A + A^T)\right)^T = \frac{1}{2}(A + A^T)^T \\ &[\because (kA)^T = kA^T] \tag{1/2} \end{aligned}$$

$$\begin{aligned} \Rightarrow P^T &= \frac{1}{2}(A^T + (A^T)^T) \\ &[\because (A + B)^T = A^T + B^T] \tag{1} \end{aligned}$$

$$\Rightarrow P^T = \frac{1}{2}(A^T + A) \quad [\because (A^T)^T = A]$$

$$\Rightarrow P^T = \frac{1}{2}(A + A^T) = P$$

[By commutative prop. of matrix over odd.]

Therefore, P is a symmetric matrix.

$$\begin{aligned} \text{Also, } Q^T &= \left(\frac{1}{2}(A - A^T)\right)^T = \frac{1}{2}(A - A^T)^T \\ &= \frac{1}{2}(A^T - (A^T)^T) = \frac{1}{2}(A^T - A) \\ &= -\frac{1}{2}(A - A^T) = -Q \end{aligned} \tag{1}$$

Therefore, Q is a skew-symmetric matrix.
Thus, $A = P + Q$, where P is a symmetric matrix and Q is a skew-symmetric matrix.
Hence A is expressible as the sum of a symmetric and a skew-symmetric matrix. (1)
Uniqueness: If possible, let $A = R + S$, where R is symmetric and S is skew-symmetric.

$$\text{Then, } A^T = (R + S)^T = R^T + S^T$$

$$\Rightarrow A^T = R - S \quad [\because R^T = R \text{ and } S^T = -S]$$

$$\text{Now, } A = R + S \text{ and } A^T = R - S$$

$$\Rightarrow R = \frac{1}{2}(A + A^T) = P$$

$$S = \frac{1}{2}(A - A^T) = Q. \tag{1}$$

Hence, A is uniquely expressible as the sum of a symmetric and a skew-symmetric matrix. Thus, we can say that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

or

$$\text{If } A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \text{ is a matrix satisfying } AA^T = 9I_3$$

, then find the values of a and b .

Ans :

$$\begin{aligned} \text{We have, } A &= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \\ \Rightarrow A^T &= \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} \end{aligned} \tag{1}$$

$$\text{Now, } AA^T = 9I_3$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$$\begin{aligned} \Rightarrow \begin{bmatrix} 9 & 0 & a+2b+4 \\ 0 & 9 & 2a+2-2b \\ a+2b+4 & 2a+2-2b & a^2+4+b^2 \end{bmatrix} &= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \end{aligned} \tag{1}$$

$$\Rightarrow a + 2b + 4 = 0$$

$$2a + 2 - 2b = 0$$

$$\text{and } a^2 + 4 + b^2 = 9 \tag{1}$$

$$\Rightarrow a + 2b + 4 = 0$$

$$a - b + 1 = 0$$

$$\text{and } a^2 + b^2 = 5$$

$$\text{Solving } a + 2b + 4 = 0$$

$$\text{and } a - b + 1 = 0$$

$$\text{We get } a = -2 \text{ and } b = -1 \tag{1}$$

34. A manufacturer produces two types of steel trunks. He has two machines *A* and *B*. The first type of trunk requires 3h on machine *A* and 3h on machine *B*. The second type of trunk requires 3h on machines *A* and 2h on machine *B*. Both machines are run daily for 18h and 15h, respectively. There is a profit of ₹30 on first type of trunk and ₹25 on the second type of trunk. How many trunks of each type should be produced and sold to make maximum profit? [6]

Ans :

Let number of trunks of first type = *x* and number of trunks of second type = *y*

Now, according to the question, required linear programming problem is maximize (1/2)

$$Z = 30x + 25y$$

Subject to the constraints are

$$3x + 3y \leq 18 \Rightarrow x + y \leq 6 \quad \dots(i)$$

and $3x + 2y \leq 15 \quad \dots(ii)$

$$x, y \geq 0 \quad \dots(iii) \quad (1/2)$$

Now, we draw the graph of above lines.

Table for $x + y = 6$,

<i>x</i>	0	6
<i>y</i>	6	0

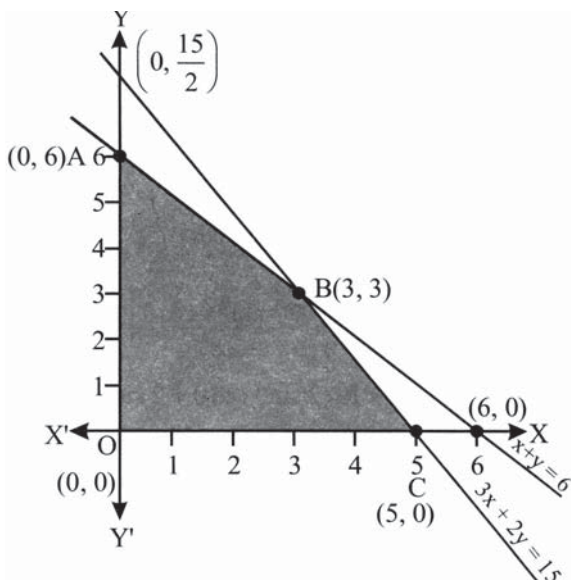
i.e., eq. (i) passes through the points (0,6) and (6, 0).

Table for $3x + 2y = 15$

<i>x</i>	5	0
<i>y</i>	0	15/2

i.e. eq., (ii) passes through the points (5, 0) and $(0, \frac{15}{2})$. (1)

∴ Graph is as follows. (1)



The region *OABC* is the feasible region. To find the corner points of the region, we solve eqs. (i) and (ii), we get

$$\begin{aligned} & 3 \times (x + y = 6) \\ & 1 \times (3x + 2y = 15) \\ \Rightarrow & \begin{array}{r} 3x + 3y = 18 \\ 3x + 2y = 15 \\ \hline y = 3 \end{array} \end{aligned} \quad (1)$$

Putting $y = 3$ in eq. (i), we get

$$x + y = 6$$

$$x + 3 = 6$$

or $x = 3$

∴ Point *B* is *B*(3,3) (1)

Now, we find value of *Z* at various corner points *O*(0,0), *A*(0,6), *B*(3,3) and *C*(5,0)

Corner Points	$Z = 30x + 25y$
0 (0, 0)	$30(0) + 25(0) = 0$
<i>A</i> (0,6)	$30(0) + 25(6) = 150$
<i>B</i> (3,3)	$30(3) + 25(3) = 90 + 75$ $= 165$ (maximum)
<i>C</i> (5,0)	$30(5) + 25(0) = 150$

(1/2)

Hence, the maximum profit ₹165 which is achieved, when 3 units of each type of trunk is produced. (1/2)

35. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the straight line $\frac{x}{a} + \frac{y}{b} = 1$. [6]

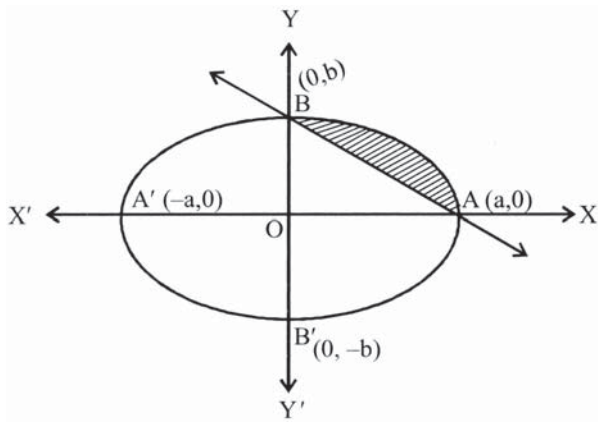
Ans :

The equation of the given curves are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots(i)$$

and $\frac{x}{a} + \frac{y}{b} = 1 \quad \dots(ii)$

Ellipse and equation of a straight line cutting *x* and *y*-axes at (*a*,0) and (0,*b*) respectively.



(2)

Required area = Area of the shaded region

$$= \int_0^a \left\{ \frac{b}{a} \sqrt{a^2 - x^2} - \frac{b}{a}(a - x) \right\} dx \quad (1)$$

$$= \frac{b}{a} \left[\int_0^a \sqrt{a^2 - x^2} dx - \int_0^a (a - x) dx \right]$$

$$= \frac{b}{a} \left[\left[\frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1} \left(\frac{x}{a} \right) \right]_0^a - \left[ax - \frac{x^2}{2} \right]_0^a \right] \quad (1\frac{1}{2})$$

$$= \left(\frac{\pi ab}{4} - \frac{1}{2} ab \right) \text{ sq. units.} \quad (1\frac{1}{2})$$

$$= \frac{1}{2} \left(\frac{\pi}{2} - 1 \right) ab \text{ sq. units.}$$

or

Evaluate $\int_a^b x dx$ using integration as limit of sum. [6]

Ans :

Let $I = \int_a^b x dx$

Comparing I with $\int_a^b f(x) dx$, we get: $a = a$,
 $b = b$, $f(x) = x$

Let $h = \frac{b-a}{n} \Rightarrow nh = b-a, n \in \mathbb{N}$

$$\therefore I = \lim_{h \rightarrow 0} h \{ f(a) + f(a+h) + f(a+2h) + \dots + f\{a+(n-1)h\} \}$$

$$= \lim_{h \rightarrow 0} h \{ a + (a+h) + (a+2h) + \dots + \{a+(n-1)h\} \} \quad [\because f(x) = x]$$

$$= \lim_{h \rightarrow 0} h \{ (a+a+a+\dots+a) + h(1+2+3+\dots+(n-1)) \}$$

$$= \lim_{h \rightarrow 0} h \left[na + h \cdot \frac{n(n-1)}{2} \right]$$

$$\left[\because [1+2+3+\dots+(n-1)] = \frac{n(n-1)}{2} \right]$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \left[nha + \frac{nh(nh-h)}{2} \right] \\ &= \lim_{h \rightarrow 0} \left[(b-a)a + \frac{(b-a)(b-a-h)}{2} \right] \\ &\quad [\because nh = (b-a)] \\ &= a(b-a) + \frac{(b-a)(b-a)}{2} \\ &= (b-a) \left[a + \frac{b-a}{2} \right] = \frac{(b-a)(b+a)}{2} \\ &= \frac{b^2 - a^2}{2} \end{aligned}$$

36. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere. [6]

Ans :

Let a cone. VAB of greatest volume be inscribed in the sphere.

Let $AOC = \theta$

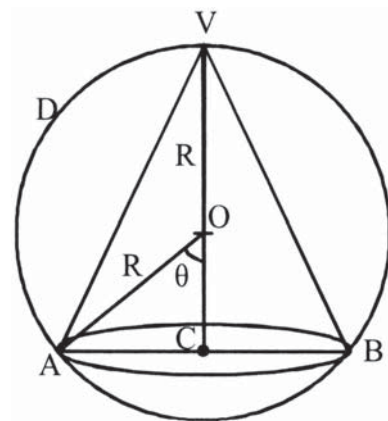
$\therefore AC$, radius of the base of the cone = $R \sin \theta$

and $VC = VO + OC = R(1 + \cos \theta)$

$$= R + R \cos \theta$$

= height of the cone

Volume of the cone, $V = \frac{1}{3} \pi (AC)^2 (VC)$



$$\begin{aligned} \Rightarrow V &= \frac{1}{3} \pi R^3 \sin^2 \theta (1 + \cos \theta) \\ &= \frac{1}{3} \pi R^3 [\sin^2 \theta + \sin^2 \theta \cos \theta] \quad (1) \end{aligned}$$

$$\therefore \frac{dV}{d\theta} = \frac{1}{3} \pi R^3 (2 \sin \theta \cos \theta + 2 \sin \theta \cos^2 \theta - \sin^3 \theta) \quad (1)$$

For maximum and minimum, we have

$$\frac{dV}{d\theta} = 0 \Rightarrow \cos \theta = \frac{1}{3} \text{ or } \cos \theta = -1 \quad (1)$$

But $\cos \theta \neq -1$ as $\cos \theta = -1$

$\Rightarrow \theta = \pi$, which is not possible. Also $\sin \theta \neq 0$

$$\Rightarrow \theta = 0 \text{ (not possible)} \therefore \cos \theta = \frac{1}{3}$$

$$\begin{aligned} \text{When } \cos \theta = \frac{1}{3}, \sin \theta &= \sqrt{1 - \cos^2 \theta} \\ &= \frac{2\sqrt{2}}{3} \end{aligned} \quad (1/2)$$

$$\left(\frac{d^2 V}{d\theta^2} \right) \text{ at } \left[\theta = \cos^{-1} \left(\frac{1}{3} \right) \right] < 0$$

$$\text{Hence } V \text{ is maximum at } \theta = \cos^{-1} \left(\frac{1}{3} \right) \quad (1)$$

$$\text{Now, } \cos \theta = \frac{1}{3}, \sin \theta = \frac{2\sqrt{2}}{3}$$

\therefore Maximum volume of cone

$$\begin{aligned} &= \frac{1}{3} \pi R^3 \left(\frac{2\sqrt{2}}{3} \right)^2 \left(1 + \frac{1}{3} \right) \\ &= \frac{8}{27} \left(\frac{4}{3} \pi R^3 \right) \end{aligned} \quad (1)$$

$$\text{Max. volume of cone} = \frac{8}{27} \times \text{volume of the sphere} \quad (1/2)$$

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