

CLASS XII (2019-20)
MATHEMATICS (041)
SAMPLE PAPER-5

Time : 3 Hours

Maximum Marks : 80

General Instructions :

- (i) All questions are compulsory.
- (ii) The questions paper consists of 36 questions divided into 4 sections A, B, C and D.
- (iii) Section A comprises of 20 questions of 1 mark each. Section B comprises of 6 questions of 2 marks each. Section C comprises of 6 questions of 4 marks each. Section D comprises of 4 questions of 6 marks each.
- (iv) There is no overall choice. However, an internal choice has been provided in three questions of 1 mark each, two questions of 6 marks each. You have to attempt only one of the alternatives in all such questions.
- (v) Use of calculators is not permitted.

Section-A

DIRECTION : (Q 1-Q 10) are multiple choice questions. Select the correct option.

1. Let R be the relation in the set $\{1, 2, 3, 4\}$ given by $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$. Then, [1]
- (a) R is reflexive and transitive but not symmetric
 - (b) R is reflexive and symmetric but not transitive
 - (c) R is symmetric and transitive but not reflexive
 - (d) R is an equivalence relation

Ans : (a) R is reflexive and transitive but not symmetric

Here, $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$

Since, $(a, a) \in R$ for every $a \in \{1, 2, 3, 4\}$.

$\therefore R$ is reflexive.

Now, since $(1, 2) \in R$ but $(2, 1) \notin R$

$\therefore R$ is not symmetric.

Also, it is observed that

$$(a, b)(b, c) \in R$$

$$(a, c) \in R$$

For all $a, b, c \in \{1, 2, 3, 4\}$

$\therefore R$ is transitive.

Hence, R is reflexive and transitive but not symmetric.

2. The normal at the point $(0, 1)$ on the curve $y = e^{2x} + x^2$ is [1]
- (a) $x + y = 0$
 - (b) $x + 2y = 2$

$$(c) \ x + 2y + 1 = 0 \quad (d) \ x - y + 1 = 0$$

Ans : (b) $x + 2y = 2$

The given curve is $y = e^{2x} + x^2$

On differentiating w.r.t. x , we get

$$\frac{dy}{dx} = 2e^{2x} + 2x$$

Slope of normal at $(0, 1)$ is

$$\frac{-1}{\left(\frac{dy}{dx}\right)_{(0,1)}} = \frac{-1}{2}$$

Equation of the normal at $(0, 1)$ is

$$y - 1 = \left(\frac{-1}{2}\right)(x - 0)$$

$$\therefore \quad x + 2y = 2$$

3. The probability of obtaining an even prime number on each die when a pair of dice is rolled, is [1]

- (a) zero
- (b) $\frac{1}{3}$
- (c) $\frac{1}{12}$
- (d) $\frac{1}{36}$

Ans : (d) $\frac{1}{36}$

When a pair of dice is rolled

Total number of outcomes = $6^2 = 36$

The only even prime is 2.

Let A be the event getting an even prime number of each die i.e., $A = \{2, 2\}$.

$$\therefore \text{Required probability} = \frac{1}{36}$$

4. If \vec{a} is a non-zero vector of magnitude $|\vec{a}|$ and λ is a non-zero scalar, then $\lambda\vec{a}$ is unit vector, if [1]
- (a) $\lambda = 1$
 - (b) $\lambda = -1$

(c) $|\vec{a}| = |\lambda|$ (d) $|\vec{a}| = \frac{1}{|\lambda|}$

Ans : (d) $|\vec{a}| = \frac{1}{|\lambda|}$

Vector λa is a unit vector.

$\therefore |\lambda \vec{a}| = 1$

$|\lambda| |\vec{a}| = 1$

$|\vec{a}| = \frac{1}{|\lambda|}$

5. $\int_{-5}^{-2} |x+2| dx$ is equal to [1]

(a) 22 (b) 29

(c) 35 (d) 15

Ans : (b) 29

Let $I = \int_{-5}^{-2} |x+2| dx$

It can be seen that $(x+2) \leq 0$ on $[-5, -2]$ and $(x+2) \geq 0$ on $[-2, 5]$.

$\therefore I = \int_{-5}^{-2} -(x+2) dx + \int_{-2}^5 (x+2) dx$
 $\left[\because \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \right]$

$I = -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^5$
 $= -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5)\right]$
 $+ \left[\frac{(5)^2}{2} + 2(5) - \frac{(-2)^2}{2} - 2(-2)\right]$
 $= -\left[2 - 4 - \frac{25}{2} + 10\right] + \left[\frac{25}{2} + 10 - 2 + 4\right]$
 $= -2 + 4 + \frac{25}{2} - 10 + \frac{25}{2} + 10 - 2 + 4$
 $= 29$

6. The number of arbitrary constants in the particular solution of differential equation of third order is [1]

(a) 3 (b) 2

(c) 1 (d) 0

Ans : (d) 0

We know that in particular solution of differential equation is free from arbitrary constant.

7. The total revenue in rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue when $x = 15$ is [1]

(a) 116 (b) 96

(c) 90 (d) 126

Ans : (d) 126

Given, $R(x) = 3x^2 + 36x + 5$

Marginal revenue, $MR = \frac{dR}{dx}$

$\therefore \frac{dR}{dx} = 6x + 36$

when $x = 15$,

$MR = 6 \times 15 + 36$
 $= 90 + 36 = 126$

Hence, the required marginal revenue is 126 at $x = 15$.

8. $\int_0^2 x\sqrt{2-x} dx$ is equal to [1]

(a) $\frac{16\sqrt{2}}{15}$ (b) $\frac{3\sqrt{2}}{5}$

(c) $\frac{4\sqrt{3}}{5}$ (d) $\frac{6\sqrt{5}}{7}$

Ans : (a) $\frac{16\sqrt{2}}{15}$

Let $I = \int_0^2 x\sqrt{2-x} dx$... (i)

Also, $I = \int_0^2 (2-x)\sqrt{2-(2-x)} dx$
 $\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$

$= \int_0^2 (2-x)\sqrt{x} dx$
 $= \int_0^2 (2x^{1/2} - x^{3/2}) dx$
 $= \left[\frac{2x^{(1/2)+1}}{(\frac{1}{2})+1} - \frac{x^{(3/2)+1}}{(\frac{3}{2})+1} \right]_0^2$
 $= \left[\frac{4}{3} x^{3/2} - \frac{2}{5} x^{5/2} \right]_0^2$
 $= \frac{4}{3} \cdot 2^{3/2} - \frac{2}{5} \cdot 2^{5/2} - 0$
 $= \frac{4}{3} 2\sqrt{2} - \frac{2}{5} 4\sqrt{2}$
 $= \left(\frac{8}{3} - \frac{8}{5}\right)\sqrt{2} = \left(\frac{40-24}{15}\right)\sqrt{2}$
 $= \frac{16\sqrt{2}}{15}$

9. For the function $f(x) = xe^x$, the point [1]

(a) $x = 0$ is a maximum

(b) $x = 0$ is a minimum

(c) $x = -1$ is a maximum

(d) $x = -1$ is a minimum

Ans : (d) $x = -1$ is a minimum

Given, $f(x) = xe^x$

$$f'(x) = e^x + xe^x$$

$$f''(x) = e^x + xe^x + e^x$$

$$= 2e^x + xe^x$$

For maxima or minima, put $f'(x) = 0$

$$e^x(1 + x) = 0$$

$$x = -1$$

At $x = -1, f''(x) > 0$

So, at $x = -1, f(x)$ is minimum.

10. $\int_0^2 \{x\} dx$ is equal to (where $\{x\}$ is fraction part of x) [1]

- (a) 2 (b) 1
(c) 5 (d) 4

Ans : (b) 1

Let $I = \int_0^2 \{x\} dx$

$$I = \int_0^2 (x - [x]) dx \quad [\because \{x\} = x - [x]]$$

$$= \int_0^2 x dx - \int_0^2 [x] dx$$

$$= \int_0^2 x dx - \int_0^1 0 dx - \int_1^2 dx$$

$$= \left[\frac{x^2}{2} \right]_0^2 - [x]_1^2$$

$$= \left(\frac{4}{2} - 0 \right) - (2 - 1)$$

$$= 2 - 0 - 2 + 1 = 1$$

DIRECTION : (Q 11-Q 15) fill in the blanks

11. A feasible solution which leads to an optimal value of the objective function is called [1]

Ans : Optimal solution

12. The range of $\cos^{-1} x$ is [1]

Ans : $[0, \pi]$

13. Every differentiable function is continuous. But a continuous function may or may not be [1]

Ans : Differentiable

or

Let $f : [a, b] \rightarrow R$ be a continuous function on $[a, b]$ and differential function in $[a, b]$. By mean value theorem, there exists atleast one c in $[a, b]$ such that $f'(c) = \dots\dots\dots$ [1]

Ans :

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

14. If A and B are square matrices such that $AB = BA$, then $(A + B)^2 = \dots\dots\dots$ [1]

Ans : $A^2 + 2AB + B^2$

or

Transpose of a column matrix is a

Ans : Row matrix

15. $\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \dots\dots\dots$ [1]

Ans : $\int_0^a f(2a - x) dx$

DIRECTION : (Q 16-Q 20) Answer the following questions.

16. If $y = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots\infty$, then prove

that $\frac{d^2 y}{dx^2} - y = 0$. [1]

Ans :

We have,

$$y = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots\infty$$

$$y = e^{-x}$$

$$\therefore \frac{dy}{dx} = -e^{-x}$$

$$\frac{d^2 y}{dx^2} = e^{-x}$$

$$\frac{d^2 y}{dx^2} = y$$

$$\frac{d^2 y}{dx^2} - y = 0$$

Hence proved.

17. If A and B are matrices of order 3 and $|A| = 5, |B| = 3$, then find $|3AB|$. [1]

Ans :

Given, $|A| = 5, |B| = 3$

We know that,

$$|kA| = k^n |A|$$

where A is matrix of order n .

$$\therefore |3AB| = 3^3 |A| \cdot |B|$$

$$= 27 \times 5 \times 3 = 405$$

18. Find the direction cosines of the line passing through the two points $(-2, 4, -5)$ and $(1, 2, 3)$. [1]

Ans :

Let $(x_1, y_1, z_1) \equiv (-2, 4, -5)$ and

$(x_2, y_2, z_2) \equiv (1, 2, 3)$
DR's of the line are

$$1 - (-2), 2 - 4, 3 - (-5) = 3, -2, 8$$

[∵ DR's of the line are $x_2 - x_1, y_2 - y_1$ and $z_2 - z_1$]

∴ DC's are

$$= \frac{3}{\sqrt{(3)^2 + (-2)^2 + (8)^2}}, \frac{-2}{\sqrt{(3)^2 + (-2)^2 + (8)^2}}, \frac{8}{\sqrt{(3)^2 + (-2)^2 + (8)^2}}$$

$$= \frac{3}{\sqrt{9 + 4 + 64}}, \frac{-2}{\sqrt{9 + 4 + 64}}, \frac{8}{\sqrt{9 + 4 + 64}}$$

$$= \frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$$

or

Find the distance of the point whose position vector is $(2\hat{i} + \hat{j} - \hat{k})$ from the plane $\vec{r}(\hat{i} - 2\hat{j} + 4\hat{k}) = 9$.

Ans :

Here, $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$
 $\vec{n} = \hat{i} - 2\hat{j} + 4\hat{k}$ and $d = 9$

So, the required distance is

$$\frac{|(2\hat{i} + \hat{j} - \hat{k})(\hat{i} - 2\hat{j} + 4\hat{k}) - 9|}{\sqrt{1 + 4 + 16}} = \frac{|2 - 2 - 4 - 9|}{\sqrt{21}}$$

$$= \frac{13}{\sqrt{21}}$$

19. Evaluate $\int_0^1 3^{x-[x]} dx$. [1]

Ans :

Let $I = \int_0^1 3^{x-[x]} dx$

$$= \int_0^1 3^{x-0} dx \quad [\because [x] = 0, x \in (0,1)]$$

$$= \left[\frac{3^x}{\log 3} \right]_0^1 \quad \left[\because \int a^x dx = \frac{a^x}{\log a} \right]$$

$$= \left[\frac{3^1}{\log 3} - \frac{3^0}{\log 3} \right] = \frac{3}{\log 3} - \frac{1}{\log 3}$$

$$= \frac{2}{\log 3}$$

20. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'number is even' and B be the event, 'number is red'. Are A and B are independent? [1]

Ans :

When a die is thrown, the sample space

$$S = \{1, 2, 3, 4, 5, 6\}$$

Also, $A =$ Number is even,

$B =$ Number is red

$$A = \{2, 4, 6\},$$

$$B = \{1, 2, 3\},$$

$$A \cap B = \{2\}$$

$$P(A) = \frac{3}{6}, \quad P(B) = \frac{3}{6},$$

$$P(A \cap B) = \frac{1}{6}$$

$$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$P(A) \times P(B) \neq P(A \cap B)$$

Thus, A and B are not independent event.

Section B

21. If \vec{a} and \vec{b} are the position vectors of A and B , respectively, find the position vector of a point C on BA produced such that $BC = 1.5BA$. [2]

Ans :

Let $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}$ and $\vec{OC} = \vec{c}$

We have, $\vec{BC} = 1.5\vec{BA}$

$$\therefore \vec{BC} = \vec{OC} - \vec{OB}$$

and $\vec{BA} = \vec{OA} - \vec{OB}$

$$\therefore \vec{OC} - \vec{OB} = 1.5(\vec{OA} - \vec{OB})$$

$$\vec{OC} - \vec{b} = 1.5(\vec{a} - \vec{b}) \quad [1]$$

$$\vec{OC} = 1.5\vec{a} - 1.5\vec{b} + \vec{b}$$

$$= 1.5\vec{a} - 0.5\vec{b}$$

$$= \frac{3}{2}\vec{a} - \frac{1}{2}\vec{b}$$

$$\therefore \vec{c} = \frac{3\vec{a} - \vec{b}}{2} \quad [1]$$

22. Show that the function $f(x)$ given by $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous at $x = 0$. [2]

Ans :

At $x = 0$,

$$\text{LHL} = \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h)$$

$$= \lim_{h \rightarrow 0} f(-h) = \lim_{h \rightarrow 0} -h \sin\left(\frac{1}{-h}\right)$$

$$= 0 \times$$

(An oscillating number between -1 and 1)

$$= 0 \quad [1/2]$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h)$$

$$= \lim_{h \rightarrow 0} f(h) = \lim_{h \rightarrow 0} h \sin \frac{1}{h}$$

$$= 0 \times$$

(An oscillating number between -1 and 1)

$$= 0 \quad [1/2]$$

and $f(0) = 0$

Thus, $f(0) = \text{LHL} = \text{RHL}$

$\therefore f(x)$ is continuous at $x = 0$. [1/2]

or

Differentiate $(\log \sin x)$ with respect to $\sqrt{\cos x}$.

Ans :

Let $u = \log \sin x$ and $v = \sqrt{\cos x}$

Then, $\frac{du}{dx} = \cot x$

and $\frac{dv}{dx} = -\frac{\sin x}{2\sqrt{\cos x}} \quad [1]$

$$\begin{aligned} \therefore \frac{du}{dv} &= \frac{\frac{du}{dx}}{\frac{dv}{dx}} = -\frac{\cot x}{\left(\frac{\sin x}{2\sqrt{\cos x}}\right)} \\ &= -2\sqrt{\cos x} \cot x \operatorname{cosec} x \end{aligned} \quad [1]$$

23. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in R . [2]

Ans :

We have, $f(x) = x^3 - 3x^2 + 3x - 100$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned} f'(x) &= 3x^2 - 6x + 3 \\ &= 3(x^2 - 2x + 1) \\ &= 3(x - 1)^2 \geq 0 \end{aligned}$$

for real values of x [1]

$\therefore f(x)$ is increasing in R . [1]

Hence proved.

24. A fair die is rolled. Consider the following events $A = \{2, 4, 6\}$, $B = \{4, 5\}$ and $C = \{3, 4, 5, 6\}$. Find

(i) $P\left(\frac{A \cup B}{C}\right)$,

(ii) $P\left(\frac{A \cap B}{C}\right)$. [2]

Ans :

Given, events are

$$A = \{2, 4, 6\}$$

$$B = \{4, 5\}$$

and

$$C = \{3, 4, 5, 6\}$$

Sample space, $S = \{1, 2, 3, 4, 5, 6\}$

Now, $A \cup B = \{2, 4, 5, 6\}$

$$A \cap B = \{4\}$$

$$A \cup B \cap C = \{2, 4, 5, 6\} \cap \{3, 4, 5, 6\}$$

$$= \{4, 5, 6\}$$

and $A \cap B \cap C = \{4\} \cap \{3, 4, 5, 6\} = \{4\}$

$\therefore n(S) = 6$

$$n(A \cup B \cap C) = 3$$

$$n(A \cap B \cap C) = 1$$

and $n(C) = 4$

(i)
$$P\left(\frac{A \cup B}{C}\right) = \frac{P(A \cup B \cap C)}{P(C)} = \frac{n(A \cup B \cap C)}{\frac{n(S)}{n(C)}} = \frac{\frac{3}{6}}{\frac{4}{6}} = \frac{3}{4} \quad [1]$$

(ii)
$$P\left(\frac{A \cap B}{C}\right) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{n(A \cap B \cap C)}{\frac{n(S)}{n(C)}} = \frac{\frac{1}{6}}{\frac{4}{6}} = \frac{1}{4} \quad [1]$$

25. Show that the determinant value of a skew-symmetric matrix of odd order is always zero. [2]

Ans :

Let A be a skew-symmetric matrix of order n .

Then, $A' = -A$

$$|A'| = |-A|$$

$$|A'| = (-1)^n |A| \quad [1]$$

$$|A'| = -|A| \quad [\because n \text{ is odd}]$$

$$|A| = -|A| \quad [\because |A'| = |A|]$$

$$2|A| = 0$$

$$|A| = 0 \quad [1]$$

Hence, determinant value of a skew-symmetric matrix of odd order is always zero.

or

Without expanding, show that

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0$$

Ans :

We have,

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 - C_2 - C_3$, we get

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta - \cot^2\theta & \cot^2\theta & 1 \\ \cot^2\theta - \operatorname{cosec}^2\theta + 1 & \operatorname{cosec}^2\theta & -1 \\ 42 - 40 - 2 & 40 & 2 \end{vmatrix} \quad [1]$$

$$\Delta = \begin{vmatrix} 0 & \cot^2\theta & 1 \\ 0 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix}$$

$$\Delta = 0 \quad \text{Hence proved.} \quad [1]$$

26. Find the minimum value of n for which $\tan^{-1} \frac{n}{\pi} > \frac{\pi}{4}$, $n \in N$. [2]

Ans :

We have,

$$\tan^{-1} \frac{n}{\pi} > \frac{\pi}{4}$$

$$\tan\left(\tan^{-1} \frac{n}{\pi}\right) > \tan \frac{\pi}{4}$$

$$\frac{n}{\pi} > 1 \quad [1]$$

$$n > \pi$$

$$n > 3.14$$

$$n = 4, 5, 6, \dots \quad [\because n \in N]$$

Hence, the minimum value of n is 4. [1]

Section C

27. Find the equation of a curve passing through the point $(0, 1)$, if the slope of the tangent to the curve at any point (x, y) is equal to the sum of the x -coordinate (abscissa) and the product of the x -coordinate and y -coordinate (ordinate) of that point. [4]

Ans :

We know that, slope of tangent to the curve

$y = f(x)$ at any point (x, y) is $\frac{dy}{dx}$.

According to the given condition, we have

$$\frac{dy}{dx} = x + xy$$

or $\frac{dy}{dx} - xy = x$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

where, $P = -x$ and $Q = x$

Now, $IF = e^{\int P dx} = e^{\int (-x) dx} = e^{-x^2/2}$ [1]

Now, the solution is given by

$$y \cdot (IF) = \int (Q \cdot IF) dx + c$$

$$y \cdot e^{-x^2/2} = \int x \cdot e^{-x^2/2} dx + c \quad [1/2]$$

Now, put $\frac{x^2}{2} = t \Rightarrow x dx = dt$

$$\therefore y \cdot e^{-x^2/2} = \int e^{-t} dt + c \quad [1]$$

$$= -e^{-t} + c = -e^{-x^2/2} + c$$

$$y = -1 + ce^{x^2/2} \quad \dots(i)$$

Since, the curve passes through the point $(0, 1)$.

We have, $1 = -1 + ce^0$

$$c = 2 \quad [1/2]$$

Hence, the required equation of curve is

$$y = -1 + 2e^{x^2/2} \quad [1]$$

28. Evaluate $\int \frac{1+x^2}{1+x^4} dx$. [4]

Ans :

Let $I = \int \frac{1+x^2}{1+x^4} dx = \int \frac{\frac{1}{x^2} + \frac{x^2}{x^2}}{\frac{1}{x^2} + \frac{x^4}{x^2}} dx$

[divide numerator and denominator by x^2]

$$= \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx \quad [1/2]$$

$$= \int \frac{1 + \frac{1}{x^2}}{\left(x^2 + \frac{1}{x^2} - 2 + 2\right)} dx$$

$$= \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 2} dx \quad [1/2]$$

Now, put $x - \frac{1}{x} = t \Rightarrow \left(1 + \frac{1}{x^2}\right)dx = dt$

$$\begin{aligned} \therefore I &= \int \frac{dt}{t^2 + 2} \\ &= \int \frac{dt}{t^2 + (\sqrt{2})^2} \\ &= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) \\ &\left[\because \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \right] \quad [1/2] \\ &= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x - \frac{1}{x}}{\sqrt{2}}\right) \\ &\quad \left[\because t = x - \frac{1}{x} \right] \\ &= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{2}x}\right) + c \quad [1/2] \end{aligned}$$

or

Evaluate $\int x \cdot (\log x)^2 dx$.

Ans :

$$\begin{aligned} \text{Let } I &= \int x \cdot (\log x)^2 dx \\ \text{On applying integration by parts, we get} \\ I &= (\log x)^2 \int x dx - \int \left[\frac{d}{dx} (\log x)^2 \cdot \int x dx \right] dx \quad [1] \\ &= (\log x)^2 \cdot \frac{x^2}{2} - \int \left[\frac{2 \log x}{x} \times \frac{x^2}{2} \right] dx \\ &= \frac{x^2}{2} (\log x)^2 - \int x \log x dx \quad [1] \end{aligned}$$

Again applying integration by parts, we get

$$\begin{aligned} I &= \frac{x^2}{2} (\log x)^2 - \left[\log x \int x dx - \int \left(\frac{d}{dx} \log x \int x dx \right) dx \right] \\ &= \frac{x^2}{2} (\log x)^2 - \left[\log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right] \quad [1] \\ &= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \cdot \log x + \int \frac{x}{2} dx \\ &= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{x^2}{4} + c \quad [1] \end{aligned}$$

29. A can hit target 4 times out of 5 times, B can hit target 3 times out of 4 times and C can hit target 2 times out of 3 times.

They fire simultaneously. Find the probability that

- (i) any two out of A, B and C will hit the target.
- (ii) none of them will hit the target.

Ans :

Here, $P(A) = P(A \text{ hit the target}) = \frac{4}{5}$

$P(B) = P(B \text{ hit the target}) = \frac{3}{4}$

and $P(C) = P(C \text{ hit the target}) = \frac{2}{3}$ [1/2]

Then, $P(\bar{A}) = 1 - P(A) = 1 - \frac{4}{5} = \frac{1}{5}$

$P(\bar{B}) = 1 - P(B) = 1 - \frac{3}{4} = \frac{1}{4}$

and $P(\bar{C}) = 1 - P(C) = 1 - \frac{2}{3} = \frac{1}{3}$ [1/2]

$$\begin{aligned} \text{(i) } P(\text{any two of them hit the target}) &= P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) \\ &\quad + P(\bar{A} \cap B \cap C) \\ &= P(A)P(B)P(\bar{C}) + P(A)P(\bar{B})P(C) \\ &\quad + P(\bar{A})P(B)P(C) \end{aligned}$$

$$\begin{aligned} &= \left(\frac{4}{5} \times \frac{3}{4} \times \frac{1}{3}\right) + \left(\frac{4}{5} \times \frac{1}{4} \times \frac{2}{3}\right) \\ &\quad + \left(\frac{1}{5} \times \frac{3}{4} \times \frac{2}{3}\right) \quad [1\frac{1}{2}] \end{aligned}$$

$$= \frac{12}{60} + \frac{8}{60} + \frac{6}{60} = \frac{26}{60} = \frac{13}{30} \quad [1/2]$$

$$\begin{aligned} \text{(ii) } P(\text{none of them hit the target}) &= P(\bar{A})P(\bar{B})P(\bar{C}) \\ &= \frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} = \frac{1}{60} \quad [1] \end{aligned}$$

or

In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that, a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that a student knows the answer given that he answered it correctly ?

Ans :

Let E_1 : the event that the student knows the answer and E_2 : the event that the student guesses the answer.

Therefore, E_1 and E_2 are mutually exclusive and exhaustive.

$$\therefore P(E_1) = \frac{3}{4}$$

and $P(E_2) = \frac{1}{4}$

Let E : the answer is correct. [1]
 The probability that the student answered correctly, given that he knows the answer, is 1 i.e. $P\left(\frac{E}{E_1}\right) = 1$

Probability that the students answered correctly, given that the guessed, is $\frac{1}{4}$ i.e., $P\left(\frac{E}{E_2}\right) = \frac{1}{4}$. [1]

By using Baye's theorem,

$$P\left(\frac{E_1}{E}\right) = \frac{P\left(\frac{E}{E_1}\right)P(E_1)}{P\left(\frac{E}{E_1}\right)P(E_1) + P\left(\frac{E}{E_2}\right)P(E_2)} \quad [1]$$

$$= \frac{1 \times \frac{3}{4}}{1 \times \frac{3}{4} + \frac{1}{4} \times \frac{1}{4}}$$

$$= \frac{\frac{3}{4}}{\frac{3}{4} + \frac{1}{16}} = \frac{\frac{3}{4}}{\frac{12+1}{16}}$$

$$= \frac{3}{4} \times \frac{16}{13}$$

$$= \frac{12}{13} \quad [1]$$

30. Let $\vec{a} = 2\hat{i} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = 4\hat{i} - 3\hat{j} + 7\hat{k}$ be three vectors. Find a vector \vec{r} which satisfies $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r} \cdot \vec{a} = 0$. [4]

Ans :

Given, $\vec{a} = 2\hat{i} + \hat{k}$,
 $\vec{b} = \hat{i} + \hat{j} + \hat{k}$,
 $\vec{c} = 4\hat{i} - 3\hat{j} + 7\hat{k}$

and for a vector \vec{r} ,

$$\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$$

and $\vec{r} \cdot \vec{a} = 0$

Now, consider $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$

$$\vec{r} \times \vec{b} - \vec{c} \times \vec{b} = \vec{0}$$

$$(\vec{r} - \vec{c}) \times \vec{b} = \vec{0} \quad [1]$$

$\Rightarrow \vec{r} - \vec{c}$ is parallel to \vec{b} .

Let $\vec{r} - \vec{c} = \lambda\vec{b}$ for some scalar λ .

$$\vec{r} = \vec{c} + \lambda\vec{b} \quad \dots(i) \quad [1]$$

Also, it is given that,

$$\vec{r} \cdot \vec{a} = 0$$

$\therefore (\vec{c} + \lambda\vec{b}) \cdot \vec{a} = 0$ [using eq.(i)]

$$\vec{c} \cdot \vec{a} + \lambda(\vec{b} \cdot \vec{a}) = 0$$

$$\lambda = \frac{\vec{c} \cdot \vec{a}}{\vec{b} \cdot \vec{a}}$$

$$= \frac{-[(4\hat{i} - 6\hat{j} + 7\hat{k}) \cdot (2\hat{i} + \hat{k})]}{[(\hat{i} + \hat{j} + \hat{k}) \cdot (2\hat{i} + \hat{k})]}$$

$$= \frac{-(8+7)}{2+1} = \frac{-15}{3} = -5 \quad [1]$$

Now, putting $\lambda = -5$ in eq. (1), we get

$$\vec{r} = \vec{c} - 5\vec{b}$$

$$= (4\hat{i} - 3\hat{j} + 7\hat{k}) - 5(\hat{i} + \hat{j} + \hat{k})$$

$$= -\hat{i} - 8\hat{j} + 2\hat{k} \quad [1]$$

31. A toy company manufactures two types of dolls, A and B . Market tests and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is almost half of that for dolls of type A . Further, the production level of dolls of type A can exceed three times the production of dolls of other type by almost 600 units. If the company makes profit of ₹12 and ₹16 per doll, respectively on dolls A and B , then how many of each should be produced weekly in order to maximise the profit? [4]

Ans :

Let the company manufactures x dolls of type A and y dolls of type B . Then, objective function is maximum profit, $Z = 12x + 16y$. Subject to the constraints

$$x + y \leq 1200, \quad y \leq \frac{x}{2}$$

$$x - 2y \geq 0 \text{ and } x \leq 3y + 600$$

$$x - 3y \leq 600 \quad [1]$$

Consider the given constraints as equations, we get

$$x + y = 1200 \quad \dots(i)$$

$$x - 2y = 0 \quad \dots(ii)$$

and $x - 3y = 600 \quad \dots(iii)$

Table for $x + y = 1200$ is

x	400	800
y	200	400

So, the line $x + y = 1200$ passes through the points $(0, 1200)$ and $(1200, 0)$.

On putting $(0, 0)$ in the inequality $x + y \leq 1200$, we get

$$0 + 0 \leq 1200$$

$$0 \leq 1200 \quad [\text{true}]$$

So, the half plane is towards the X -axis.

Table for $x - 3y = 600$ is

x	0	600
y	-200	0

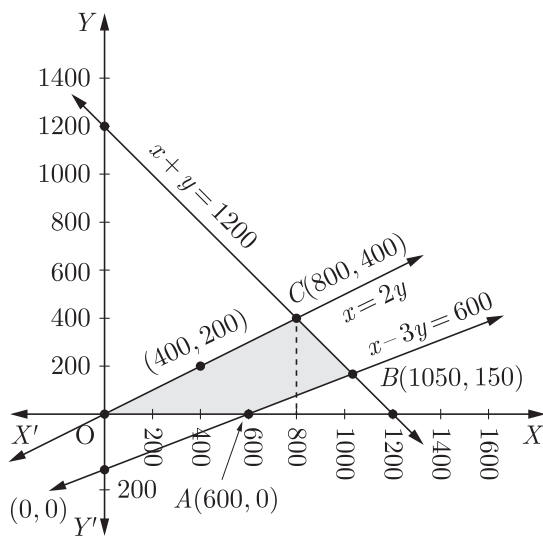
So, the line $x - 3y = 600$ passes through the points $(0, -200)$ and $(600, 0)$.

On putting $(0, 0)$ in the inequality $x - 3y \leq 600$, we get

$$0 - 3(0) \leq 600$$

$$0 \leq 600 \quad [\text{true}]$$

So, the half plane is towards the origin.
Now, intersection point of eqs.(i) and (ii) is $C(800, 400)$ and intersection point of eqs.(iii) and (i) is $B(1050, 150)$.



[1]

Now, plotting the graph of equations, the shaded portion $OABC$ represents the feasible region which is bounded and coordinates of the corner points are $O(0,0)$, $A(600,0)$, $B(1050,150)$ and $C(800,400)$.

Now, the value of Z at each corner point is given below

Corner points	$Z = 12x + 16y$
$O(0,0)$	$Z = 12(0) + 16(0)$ $= 0 + 0 = 0$
$A(600,0)$	$Z = 12(600) + 16(0)$ $= 7200$
$B(1050,150)$	$Z = 12(1050) + 16(150)$ $= 12600 + 2400 = 15000$
$C(800,400)$	$Z = 12(800) + 16(400)$ $= 9600 + 6400 = 16000$ (maximum)

\therefore Maximum value of Z is 16000 at the point $C(800,400)$.

Hence, maximum profit is ₹ 16000 when 800 dolls of type A and 400 dolls of type B are produced. [1]

or

If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ and $\vec{a} \neq \vec{0}$, then prove that $\vec{b} = \vec{c}$.

Ans :

Given,

$$\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} \text{ and } \vec{a} \neq \vec{0}$$

$$\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} = \vec{0} \text{ and } \vec{a} \neq \vec{0}$$

$$\vec{a}(\vec{b} - \vec{c}) = \vec{0} \text{ and } \vec{a} \neq \vec{0}$$

$$\vec{b} - \vec{c} = \vec{0} \text{ or } \vec{a} \perp (\vec{b} - \vec{c})$$

$$\vec{b} = \vec{c} \text{ or } \vec{a} \perp (\vec{b} - \vec{c}) \dots (i) 1\frac{1}{2}$$

Also, given $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ and $\vec{a} \neq \vec{0}$

$$\vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{0} \text{ and } \vec{a} \neq \vec{0}$$

$$\vec{a} \times (\vec{b} - \vec{c}) = \vec{0} \text{ and } \vec{a} \neq \vec{0}$$

$$\vec{b} - \vec{c} = \vec{0} \text{ or } \vec{a} \parallel (\vec{b} - \vec{c})$$

$$\vec{b} = \vec{c} \text{ or } \vec{a} \parallel (\vec{b} - \vec{c}) \dots (ii) 1\frac{1}{2}$$

From Eqs. (i) and (ii), we get

$$\vec{b} = \vec{c} \text{ [} \vec{a} \perp (\vec{b} - \vec{c}) \text{ and } \vec{a} \parallel (\vec{b} - \vec{c}) \text{ cannot hold simultaneously]}$$

Hence proved

32. Show that $f: R - (-1) \rightarrow R - \{1\}$ given by $f(x) = \frac{x}{x+1}$ is invertible. Also, find f^{-1} . [4]

Ans :

In order to prove the invertibility of $f(x)$, it is sufficient to show that it is a bijection.

f is one-one

For any $x, y \in R - \{-1\}$

$$f(x) = f(y)$$

$$\frac{x}{x+1} = \frac{y}{y+1}$$

So, f is one-one. [1]

f is onto

Let $y \in R - \{-1\}$

Then, $f(x) = y$

$$\frac{x}{x+1} = y$$

$$x = \frac{y}{1-y} \quad [1]$$

Clearly, $x \in R$ for all $y \in R - \{1\}$.

Also, $x \neq 1$. Because,

$$x = -1$$

$$\frac{y}{1-y} = -1$$

$$y = -1 + y$$

which is not possible.

Thus, for each $y \in R - \{1\}$ there exists

$$x = \frac{y}{1-y} \in R - \{-1\}$$
 such that

$$\begin{aligned} f(x) &= \frac{x}{x+1} \\ &= \frac{\frac{y}{1-y}}{\frac{y}{1-y} + 1} = y \end{aligned}$$

So, f is onto. [1]

Thus, f is both one-one and onto.

Consequently it is invertible.

Now, $f \circ f^{-1}(x) = x$ for all $x \in R - \{1\}$

$$f(f^{-1}(x)) = x$$

$$\frac{f^{-1}(x)}{f^{-1}(x)+1} = x$$

$$f^{-1}(x) = \frac{x}{1-x} \text{ for all } x \in R - \{1\}$$
 [1]

Section D

33. Show that the normal at any point θ to the curve $x = a \cos \theta + a \theta \sin \theta$ and $y = a \sin \theta - a \theta \cos \theta$ is at a constant distance from the origin. [6]

Ans :

Given, curves are

$$x = a \cos \theta + a \theta \sin \theta \quad \dots(i)$$

$$\text{and } y = a \sin \theta - a \theta \cos \theta \quad \dots(ii)$$

On differentiating both sides of eq.(i) w.r.t. θ , we get

$$\begin{aligned} \frac{dx}{d\theta} &= -a \sin \theta + a(\theta \cos \theta + \sin \theta) \\ &= -a \sin \theta + a \theta \cos \theta + a \sin \theta \\ &= a \theta \cos \theta \end{aligned} \quad [1/2]$$

On differentiating both sides of eq.(ii) w.r.t. θ , we get

$$\begin{aligned} \frac{dy}{d\theta} &= a \cos \theta - a[\theta(-\sin \theta) + \cos \theta] \\ &= a \cos \theta + a \theta \sin \theta - a \cos \theta \\ &= a \theta \sin \theta \end{aligned} \quad [1/2]$$

\therefore Slope of the tangent at θ ,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{a \theta \sin \theta}{a \theta \cos \theta} \\ &= \tan \theta \end{aligned} \quad [1]$$

$$\begin{aligned} \text{Then, slope of the normal at } \theta &= \frac{-1}{\frac{dy}{dx}} \\ &= \frac{-1}{\tan \theta} \\ &= -\cot \theta \quad [1] \end{aligned}$$

Thus, the equation of the normal at a given point (x, y) is given by

$$\begin{aligned} y - (a \sin \theta - a \theta \cos \theta) &= -\cot \theta [x - (a \cos \theta + a \theta \sin \theta)] \\ y - (a \sin \theta - a \theta \cos \theta) &= \frac{-\cos \theta}{\sin \theta} [x - (a \cos \theta + a \theta \sin \theta)] \end{aligned}$$

$$\begin{aligned} y \sin \theta - a \sin^2 \theta + a \theta \sin \theta \cos \theta &= -x \cos \theta + a \cos^2 \theta + a \theta \sin \theta \cos \theta \quad [1] \\ x \cos \theta + y \sin \theta &= a(\sin^2 \theta + \cos^2 \theta) \end{aligned}$$

$$\begin{aligned} x \cos \theta + y \sin \theta &= a \quad [\because \sin^2 x + \cos^2 x = 1] \\ x \cos \theta + y \sin \theta - a &= 0 \quad [1] \end{aligned}$$

Now, the perpendicular distance of the normal from the origin = $\frac{|0+0-a|}{\sqrt{\cos^2 \theta + \sin^2 \theta}} = \frac{|-a|}{\sqrt{1}} = a$, which is a constant.

Hence, the normal at point θ to the given curve is at a constant distance from the origin. [1]

or

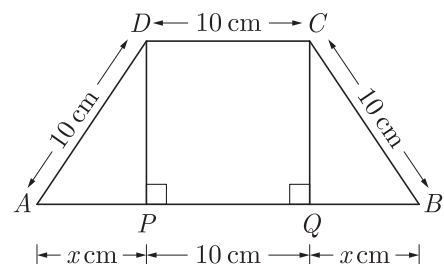
If the length of three sides of a trapezium other than base are equal to 10 cm, find the area of the trapezium when it is maximum.

Ans :

Let $ABCD$ be a trapezium such that DC is parallel to AB and $AD = 10 \text{ cm} = DC = BC$. Now, draw perpendiculars DP and CQ from D and C , on AB , respectively.

$$\therefore \Delta APD \sim \Delta BQC$$

Therefore, $PA = QB = x \text{ cm}$



In right angled ΔAPD , we have

$$\begin{aligned} AD^2 &= AP^2 + PD^2 \\ &\text{[by Pythagoras theorem]} \end{aligned}$$

$$PD^2 = AD^2 - AP^2$$

$$PD = \sqrt{AD^2 - AP^2}$$

$$= \sqrt{100 - x^2} \text{ cm} \quad [1]$$

Similarly, in ΔBQC ,

$$QC = \sqrt{100 - x^2} \text{ cm}$$

If A denotes the area of the trapezium $ABCD$, then

$$A = f(x) = \frac{1}{2} (\text{Sum of parallel sides}) \times \text{Height}$$

$$f(x) = \frac{1}{2}(AB + DC) \times PD$$

$$= \frac{1}{2}[(2x + 10) + 10] \times \sqrt{100 - x^2}$$

$$= (x + 10)\sqrt{100 - x^2} \quad \dots(i) \quad [1]$$

On differentiating both sides of eq. (i) w.r.t. x , we get

$$f'(x) = 1\sqrt{100 - x^2} + (x + 10)\left(\frac{-2x}{2\sqrt{100 - x^2}}\right)$$

$$f'(x) = \frac{(100 - x^2) - x^2 - 10x}{\sqrt{100 - x^2}}$$

$$= \frac{100 - 2x^2 - 10x}{\sqrt{100 - x^2}}$$

Again, differentiating both sides w.r.t. x , we get

$$f''(x) = \frac{\left[\begin{array}{l} \sqrt{100 - x^2}(-4x - 10) \\ -(100 - 2x^2 - 10x)\left(\frac{-2x}{2\sqrt{100 - x^2}}\right) \end{array} \right]}{(100 - x^2)}$$

$$= \frac{\left[\begin{array}{l} (100 - x^2)(-4x - 10) \\ + (100x - 2x^3 - 10x^2) \end{array} \right]}{(100 - x^2)\sqrt{100 - x^2}} \quad [2]$$

$$= \frac{\left[\begin{array}{l} (-400x - 1000 + 4x^3 + 10x^2) \\ + (100x - 2x^3 - 10x^2) \end{array} \right]}{(100 - x^2)\sqrt{100 - x^2}} \quad \dots(ii)$$

For maxima or minima, put

$$f'(x) = 0$$

$$\frac{100 - 2x^2 - 10x}{\sqrt{100 - x^2}} = 0$$

$$100 - 2x^2 - 10x = 0$$

$$2x^2 + 10x - 100 = 0$$

$$2(x^2 + 5x - 50) = 0$$

$$x^2 + 5x - 50 = 0$$

$$x^2 + 10x - 5x - 50 = 0$$

$$x(x + 10) - 5(x + 10) = 0$$

$$(x + 10)(x - 5) = 0$$

$$\therefore x = -10, x = 5$$

$$x = 5$$

$[\because x$ represents distance, so it cannot be negative] [1]

On putting $x = 5$ in eq.(ii), we get

$$[f''(x)]_{\text{at } x=5} = \frac{\left(\begin{array}{l} [-400(5) - 1000 + 4(5)^3 + 10(5)^2] \\ + [100(5) - 2(5)^3 - 10(5)^2] \end{array} \right)}{(100 - 25)\sqrt{100 - 25}}$$

$$= \frac{(-3000 + 500 + 250) + (500 - 250 - 250)}{75\sqrt{75}}$$

$$= \frac{-2250}{75\sqrt{75}} = \frac{-30}{\sqrt{75}} < 0$$

Hence, the area of trapezium is maximum, when $x = 5$ and the area is given by

$$f(5) = (5 + 10)\sqrt{100 - 25} = 15\sqrt{75}$$

$$= 75\sqrt{3} \text{ cm}^2 \quad [1]$$

34. Find the image of the point $(1, 6, 3)$ on the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$. Also, write the equation of the line joining the given point and its image and find the length of segment joining the given point and its image. [6]

Ans :

Given, point $P = (1, 6, 3)$ and equation of line AB is $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

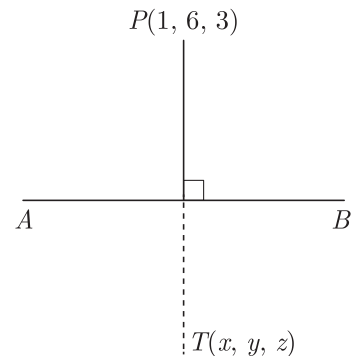
Let $T(x, y, z)$ be the image of the point $P(1, 6, 3)$ and Q be the foot of perpendicular PQ on line AB .

$$\text{Then, } \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda \text{ (say)} \quad \dots(i)$$

$$x = \lambda, \quad y = 2\lambda + 1, \quad z = 3\lambda + 2$$

Then, coordinates of

$$Q = (\lambda, 2\lambda + 1, 3\lambda + 2) \quad \dots(ii)$$



[1]

Now, DR 's of line

$$PQ = (\lambda - 1, 2\lambda + 1 - 6, 3\lambda + 2 - 3) \\ = (\lambda - 1, 2\lambda - 5, 3\lambda - 1) \quad \dots(iii)$$

Since, $PQ \perp AB$.

$$\therefore a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

where, $a_1 = \lambda - 1$, $b_1 = 2\lambda - 5$, $c_1 = 3\lambda - 1$
and $a_2 = 1$, $b_2 = 2$, $c_2 = 3$

$$\therefore 1(\lambda - 1) + 2(2\lambda - 5) + 3(3\lambda - 1) = 0 \quad [1] \\ \lambda - 1 + 4\lambda - 10 + 9\lambda - 3 = 0 \\ 14\lambda - 14 = 0 \\ \lambda = 1 \quad [1]$$

On putting $\lambda = 1$ in eq.(ii), we get

$$\text{Coordinates of } Q = (1, 2 + 1, 3 + 2) \\ = (1, 3, 5)$$

Also, Q is the mid-point of PT .

$$\therefore Q = \left(\frac{x+1}{2}, \frac{y+6}{2}, \frac{z+3}{2} \right) \quad [1]$$

$$[\because \text{mid-point of } (x_1, y_1, z_1) \text{ and } (x_2, y_2, z_2) \\ = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)]$$

But $Q = (1, 3, 5)$

$$\therefore \left(\frac{x+1}{2}, \frac{y+6}{2}, \frac{z+3}{2} \right) = (1, 3, 5) \\ \frac{x+1}{2} = 1, \frac{y+6}{2} = 3, \frac{z+3}{2} = 5$$

$$x = 2 - 1, y = 6 - 6, z = 10 - 3$$

$$x = 1, y = 0 \text{ and } z = 7$$

So, coordinates of $T = (x, y, z) = (1, 0, 7)$

Hence, coordinates of image of the point $P(1, 6, 3)$ is $T(1, 0, 7)$. [1]

Length of the segment joining P and T ,

$$PT = \sqrt{(1-1)^2 + (0-6)^2 + (7-3)^2} \\ = \sqrt{0 + 36 + 16} \\ = \sqrt{52} = 2\sqrt{13} \text{ units}$$

DR 's of line $PT = (0, -6, 4)$

$$DR\text{'s of line } PQ = [1 - 1, 2(1) - 5, 3(1) - 1] \\ \text{[from eq. (iii)]} \\ = (0, -3, 2)$$

Hence, the equation of line PT is

$$\frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2} \quad [1]$$

or

Find the foot of the perpendicular from the point $(0, 2, 3)$ on the line $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$

Also, find the length of the perpendicular.

Ans :

Let L be the foot of the perpendicular drawn from the point $P(0, 2, 3)$ to the given line. The coordinates of a general point of the line $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$ are given by

$$\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3} = \lambda$$

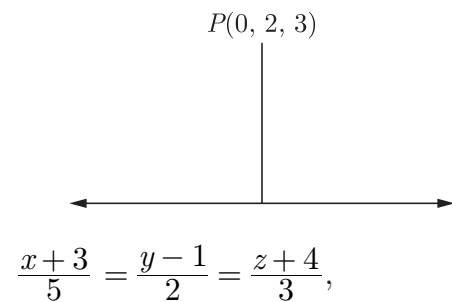
or

$$x = 5\lambda - 3,$$

$$y = 2\lambda + 1,$$

$$z = 3\lambda - 4$$

Let the coordinates of L be $(5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$. Therefore direction ratios of PL are proportional to $5\lambda - 3 - 0, 2\lambda + 1 - 2, 3\lambda - 4 - 3$ i.e., $5\lambda - 3, 2\lambda - 1, 3\lambda - 7$. (2)



$$L(5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$$

Direction ratios of the given line are proportional to 5, 2, 3. But, PL is perpendicular to the given line.

$$\therefore 5(5\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 7) = 0 \\ \lambda = 1$$

(2)

Putting $\lambda = 1$ in $(5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$, the coordinates of L are $(2, 3, -1)$.

$$\therefore PL = \sqrt{(2-0)^2 + (3-2)^2 + (-1-3)^2} \\ = \sqrt{21} \text{ units}$$

Hence, length of the perpendicular from P on the given line is $PL = \sqrt{21}$ units. (2)

35. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$. [6]

Ans :

Given equations of ellipse and the straight line are

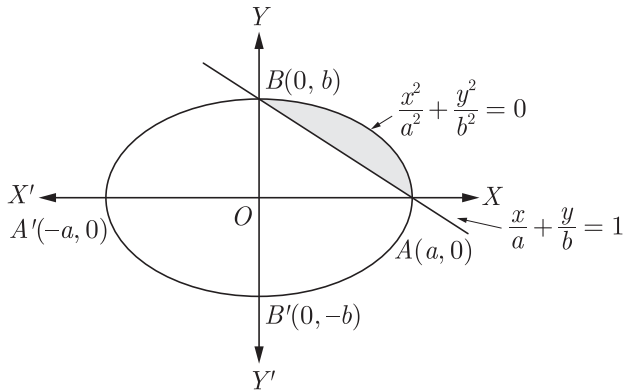
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots(i)$$

and $\frac{x}{a} + \frac{y}{b} = 1 \quad \dots(ii)$

Ellipse with eq.(i) has vertices $(\pm a, 0)$ and $(0, \pm b)$ and centre $(0, 0)$. While the line with eq.(ii) has x -intercept a and y -intercept b .

So, line passes through the points $(a, 0)$ and $(0, b)$. [1]

\therefore Graph of the above region is given below



Clearly, points of intersection are $A(a, 0)$ and $B(0, b)$.

\therefore Required area $= \int_0^a [y_{(\text{ellipse})} - y_{(\text{line})}] dx$ [1]

Now, equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\begin{aligned} \frac{y^2}{b^2} &= 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} \\ y^2 &= \frac{b^2}{a^2}(a^2 - x^2) \\ y &= \frac{b}{a}\sqrt{a^2 - x^2} \quad \dots(iv) \end{aligned}$$

and equation of line is $\frac{x}{a} + \frac{y}{b} = 1$

$$\begin{aligned} \frac{y}{b} &= 1 - \frac{x}{a} = \frac{a - x}{a} \\ y &= \frac{b}{a}(a - x) \quad \dots(v) \quad [1] \end{aligned}$$

Hence, from eqs.(iii), (iv) and (v),

Required area

$$\begin{aligned} &= \int_0^a \left[\frac{b}{a}\sqrt{a^2 - x^2} - \frac{b}{a}(a - x) \right] dx \\ &= \frac{b}{a} \int_0^a \sqrt{a^2 - x^2} dx \\ &\quad - \frac{b}{a} \int_0^a (a - x) dx \quad [1] \end{aligned}$$

We know that,

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

$$\begin{aligned} \therefore \text{Area} &= \frac{b}{a} \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]_0^a \\ &\quad - \frac{b}{a} \left[ax - \frac{x^2}{2} \right]_0^a \\ &= \frac{b}{a} \left[\frac{a^2}{2}\sin^{-1}1 \right] - \frac{b}{a} \left[a^2 - \frac{a^2}{2} \right] \quad [1] \\ &= \frac{ba^2}{2a}\sin^{-1}\left(\sin\frac{\pi}{2}\right) - \frac{b}{a}\left(\frac{a^2}{2}\right) \\ &\left[\because 1 = \sin\frac{\pi}{2} \Rightarrow \sin^{-1}1 = \sin^{-1}\left(\sin\frac{\pi}{2}\right) \right] \\ &= \left(\frac{ba}{2} \times \frac{\pi}{2}\right) - \frac{ab}{2} = \frac{\pi ab}{4} - \frac{ab}{2} \\ &= \left(\frac{\pi}{4} - \frac{1}{2}\right)ab \text{ sq units} \quad [1] \end{aligned}$$

36. Solve the following system of equations by matrix method, where $x \neq 0$, $y \neq 0$ and $z \neq 0$.

$$\frac{2}{x} - \frac{3}{y} + \frac{3}{z} = 10,$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 10$$

and

$$\frac{3}{x} - \frac{1}{y} + \frac{2}{z} = 13$$

[6]

Ans :

Given system of equations can be written in matrix form as

$$AX = B \quad \dots(i)$$

where $A = \begin{bmatrix} 2 & -3 & 3 \\ 1 & 1 & 1 \\ 3 & -1 & 2 \end{bmatrix}$,

$$X = \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix}$$

and $B = \begin{bmatrix} 10 \\ 10 \\ 13 \end{bmatrix}$

$$\begin{aligned} \text{Now, } |A| &= \begin{vmatrix} 2 & -3 & 3 \\ 1 & 1 & 1 \\ 3 & -1 & 2 \end{vmatrix} \\ &= 2(2 + 1) + 3(2 - 3) + 3(-1 - 3) \\ &\quad \text{[expanding along } R_1] \\ &= 6 - 3 - 12 = -9 \neq 0 \end{aligned}$$

$\therefore A^{-1}$ exists. [1/2]

Now, cofactors of elements of $|A|$ are

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 2 + 1 = 3$$

$$\begin{aligned}
 C_{12} &= (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -(2 - 3) = 1 \\
 C_{13} &= (-1)^{1+3} \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -1 - 3 = -4 \\
 C_{21} &= (-1)^{2+1} \begin{vmatrix} -3 & 3 \\ -1 & 2 \end{vmatrix} = -(-6 + 3) = 3 \\
 C_{22} &= (-1)^{2+2} \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 4 - 9 = -5 \\
 C_{23} &= (-1)^{2+3} \begin{vmatrix} 2 & -3 \\ 3 & -1 \end{vmatrix} \\
 &= -(-2 + 9) = -7 \\
 C_{31} &= (-1)^{3+1} \begin{vmatrix} -3 & 3 \\ 1 & 1 \end{vmatrix} = -3 - 3 = -6 \\
 C_{32} &= (-1)^{3+2} \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -(2 - 3) = 1 \\
 C_{33} &= (-1)^{3+3} \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = 2 + 3 = 5
 \end{aligned}$$

[1½]

$$\begin{aligned}
 &= \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \quad [\text{dividing each element by} \\
 & -9] \\
 \therefore \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix} &= \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \quad [1]
 \end{aligned}$$

On comparing the corresponding elements, we get

$$\frac{1}{x} = 2 \Rightarrow x = \frac{1}{2}$$

$$\frac{1}{y} = 3 \Rightarrow y = \frac{1}{3}$$

$$\frac{1}{z} = 5 \Rightarrow z = \frac{1}{5}$$

and

Hence, $x = \frac{1}{2}, y = \frac{1}{3}, z = \frac{1}{5}$ [1]

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$$\begin{aligned}
 \therefore \text{adj}(A) &= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T \\
 &= \begin{bmatrix} 3 & 1 & -4 \\ 3 & -5 & -7 \\ -6 & 1 & 5 \end{bmatrix}^T \\
 &= \begin{bmatrix} 3 & 3 & -6 \\ 1 & -5 & 1 \\ -4 & -7 & 5 \end{bmatrix} \quad [1]
 \end{aligned}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

$$A^{-1} = \frac{1}{-9} \begin{bmatrix} 3 & 3 & -6 \\ 1 & -5 & 1 \\ -4 & -7 & 5 \end{bmatrix}$$

Now, eq.(1) can be written as $X = A^{-1}B$.

$$\begin{aligned}
 \text{i.e., } \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix} &= \frac{1}{-9} \begin{bmatrix} 3 & 3 & -6 \\ 1 & -5 & 1 \\ -4 & -7 & 5 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 13 \end{bmatrix} \quad [1] \\
 &= \frac{-1}{9} \begin{bmatrix} 30 + 30 - 78 \\ 10 - 50 + 13 \\ -40 - 70 + 65 \end{bmatrix} \\
 &= \frac{-1}{9} \begin{bmatrix} -18 \\ -27 \\ -45 \end{bmatrix}
 \end{aligned}$$

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