Ch 7 - Alternating Current

AC Voltage applied to a resistor:

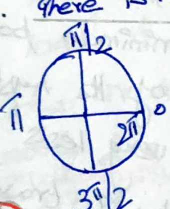
$$\boxed{V = V_m \sin \omega t}$$

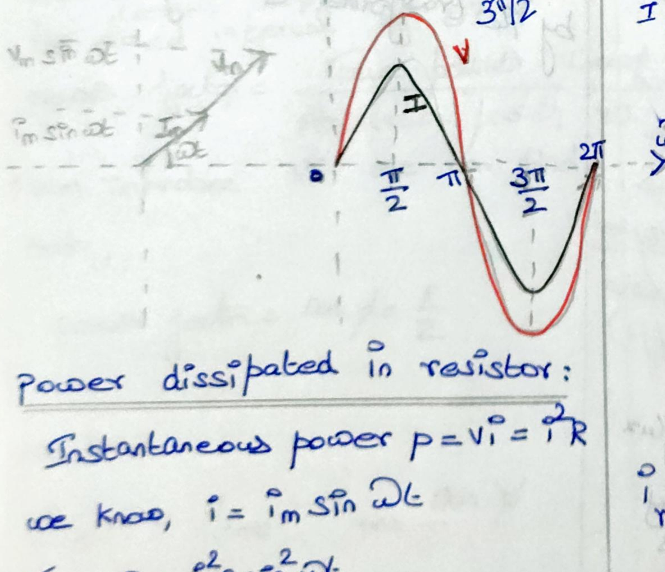
$$V = iR = V_m \sin \omega t$$

$$\therefore i = \frac{V_m}{R} \sin \omega t$$

$$\boxed{i = i_m \sin \omega t.}$$

Phase Relationship: A crucial characteristic for a purely resistive AC circuit is that current and voltage are in same phase. (means) they reach their max, min and zero values simultaneously. There is no phase diff. btn. them.





Power dissipated in resistor:

Instantaneous power $p = vi = i^2 R$

we know, $i = i_m \sin \omega t$

Then $P = i_m^2 R \sin^2 \omega t$

Average power over one cycle,

$$\bar{P} = \langle P \rangle = i_m^2 R \langle \sin^2 \omega t \rangle$$

$$\bar{P} = i_m^2 R \left\langle \frac{1 - \cos 2\omega t}{2} \right\rangle$$

$$= \frac{i_m^2 R}{2} [1 - \langle \cos 2\omega t \rangle]$$

1 → Avg. is const.

$\cos 2\omega t$ → Avg. of $\cos 2\omega t$ over one period is zero. because, (a complete cycle) $\cos 2\theta$ oscillates btn. -1 & $+1$ with a period of π. Over one complete interval of π, the +ve & -ve areas cancel each other.

$$\bar{P} = \frac{i_m^2}{2} R$$

RMS Current (effective current)

says how much DC current is there in AC current.

$$i = i_m \sin \omega t$$

$$i^2 = i_m^2 \sin^2 \omega t$$

$$i_{rms} = I = \sqrt{\langle i^2 \rangle} \quad \text{avg. of } i^2$$

$$I = \sqrt{i_m^2 \sin^2 \omega t}$$

$$I = [i_m^2 \langle \sin^2 \omega t \rangle]^{1/2}$$

$$= \left[i_m^2 \left\langle \frac{1 - \cos 2\omega t}{2} \right\rangle \right]^{1/2}$$

$$= \left[\frac{i_m^2}{2} \right]^{1/2}$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$\langle \sin^2\theta \rangle = \frac{1 - \cos 2\theta}{2}$

$= \frac{1}{2} - \frac{1}{2}\langle \cos 2\theta \rangle$

The

$$I = \frac{i_m}{\sqrt{2}}$$

$$i_{rms} = I = \frac{i_m}{\sqrt{2}}$$

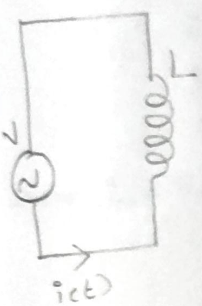
$$\text{III}^{ly} \quad V_{rms} = V = \frac{V_m}{\sqrt{2}}$$

$$P = I^2 R = VI$$

Representation of alternating current and alternating voltage as vectors (Phasors) with phase angle btn. them is called a phasor diagram.

A phasor is a vector, which rotates about its origin with angular frequency / speed ω.

Ac voltage applied to an Inductor:



when Ac voltage is supplied to an inductor, at a particular instant, the current in the circuit is 'i' at time 't'.

Now, suppose the supply voltage $\boxed{V = V_m \sin \omega t.}$

If current 'i' is flowing through the inductor having self inductance L, and the current is changing, then back emf $\mathcal{E} = -L\frac{dI}{dt}$

(as we discussed in self-inductance)

If we apply kirchoff's second rule to this particular loop, then

$$V - L\frac{di}{dt} = 0$$

where, $L\frac{di}{dt} = V$

$$L\frac{di}{dt} = V_m \sin \omega t$$

$$di = \frac{V_m}{L} \sin \omega t \cdot dt$$

Integrating this equation,

$$i = \frac{V_m}{L}\left(-\frac{\cos \omega t}{\omega}\right)$$

$$i = \frac{V_m}{\omega L}(-\cos \omega t)$$

$$\boxed{i = i_m(-\cos \omega t)}$$

$$i = i_m\left(\sin \omega t - \frac{\pi}{2}\right)$$

where, $i_m = \frac{V_m}{\omega L} = \frac{V_m}{X_L}$

$X_L = \omega L =$ Inductive reactance.

phase Relationship: current lags behind the voltage by $\pi/2$ (or) voltage leads current by $\pi/2$

Instantaneous power supplied to Inductor:

$$P = Vi = (V_m \sin \omega t)(-i_m \cos \omega t)$$

$$P = -V_m i_m (\sin \omega t \cos \omega t)$$

Multiplying and dividing the above equation by 2,

$$P = \frac{-V_m i_m}{2}(2 \sin \omega t \cos \omega t)$$

$$P = -\frac{V_m i_m}{2}(\sin 2\omega t)$$

Above equation represents instantaneous power at any time 't'.

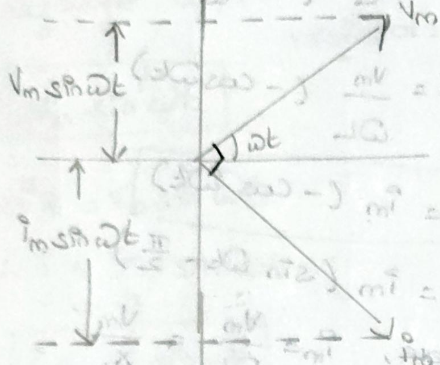
Average power over one complete cycle

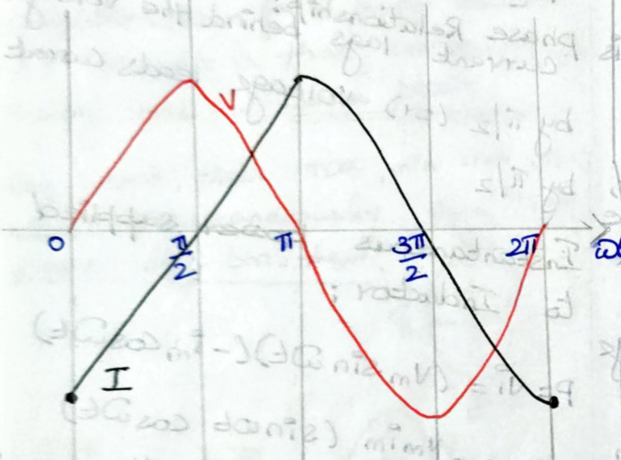
$$\bar{P} = \underbrace{-\frac{V_m i_m}{2}}_{\text{constant}} \langle \sin 2\omega t \rangle$$

↳ Average of $\sin 2\omega t$ is zero.

$\therefore \bar{P} = 0.$

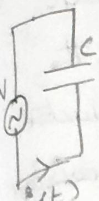
when inductor connected to Ac supply, the average power over one cycle is equal to zero.





Vatage, current Vs ωt graph
(Inductor).

Ac Voltage applied to a capacitor:



$$V = V_m \sin \omega t$$

Suppose, charge on capacitor at instant 't' is q, than $V = \frac{q}{c} = V_m \sin \omega t$

$$\therefore q = V_m c (\sin \omega t)$$

where $V \rightarrow$ is the potential difference across the two plates of a capacitor.

Instantaneous Current $i = \frac{dq}{dt} = \frac{d}{dt} V_m c (\sin \omega t)$

$$i = V_m c (\omega \cos \omega t)$$

$$i = V_m \omega c \cos \omega t$$

$$i = i_m \cos \omega t$$

$$i = i_m \sin(\omega t + \frac{\pi}{2})$$

where $i_m = V_m \omega c = \frac{V_m}{\frac{1}{\omega c}} = \frac{V_m}{X_c}$

where $X_c = \frac{1}{\omega c}$ (capacitive reactance) ↳ Resistance offered by the capacitor to the circuit.

phase Relationship:

At any time t, comparing V and i, voltage lags behind current by $\pi/2$ (or) current leads Voltage by $\pi/2$.

Instantaneous power supplied to capacitor:

$$P = Vi = (V_m \sin \omega t)(i_m \cos \omega t)$$

$$P = V_m i_m (\sin \omega t \cos \omega t)$$

Multiplying and dividing the above eqn. by 2,

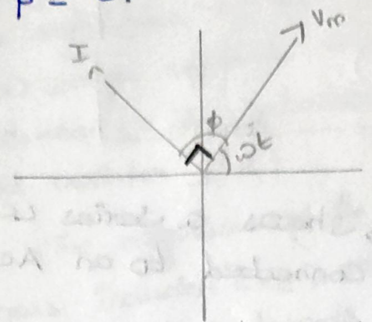
$$P = \frac{V_m i_m}{2} (2 \sin \omega t \cos \omega t)$$

$$P = \frac{V_m i_m}{2} (\sin 2\omega t)$$

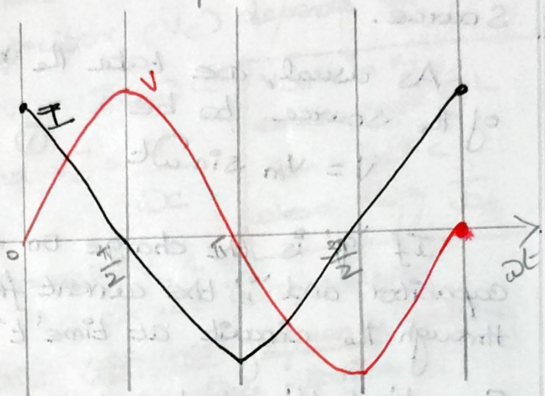
Average power over one cycle,

$$\bar{P} = \frac{V_m i_m}{2} \langle \sin 2\omega t \rangle$$

$$\bar{P} = 0.$$

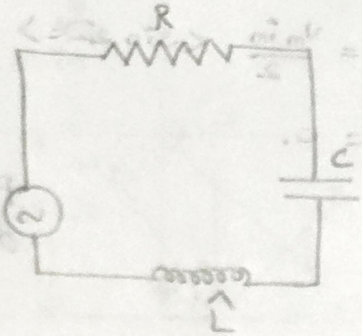


phasor diagram



Voltage, current Vs ωt graph

Ac Voltage applied to a series LCR circuit :



Figure, shows a series LCR circuit connected to an Ac source.

As usual, we take the voltage of the source to be

$$V = V_m \sin \omega t \quad (1)$$

If 'q' is the charge on the capacitor and 'i' the current flowing through the circuit at time 't'

From Kirchoff's loop rule :
(The summation of potential drops across this components is equal to supply voltage)

$$L\frac{di}{dt} + iR + \frac{q}{c} = V$$

Phasor-diagram Solution :

In the given circuit, we see that the resistor, inductor and capacitor are in series. Therefore, Ac current in each element is the same at any time, having same amplitude and phase. Let it be

$$i = i_m \sin(\omega t + \phi) \quad (2)$$

where $\phi \rightarrow$ phase difference btn. voltage and current in the circuit.

Let I be the phasor representing the current in the circuit.

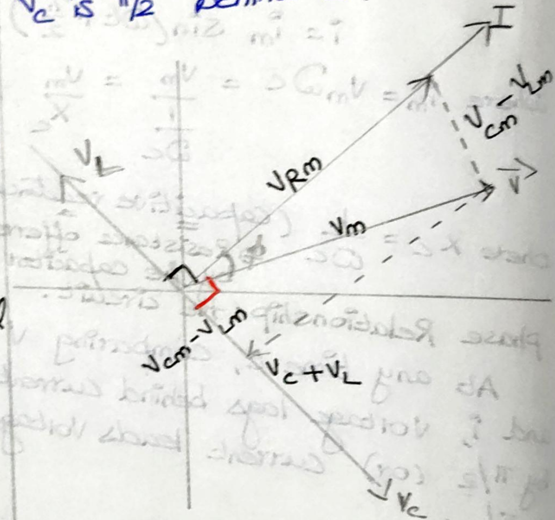
Let $V_L \rightarrow$ voltage across the inductor

$V_R \rightarrow$ voltage across the resistor

$V_c \rightarrow$ voltage across the capacitor

$V \rightarrow$ voltage across the source

We know, in case of resistor, V_R is parallel to I, in case of inductor, V_L is $\pi/2$ ahead of I and in case of capacitor, V_c is $\pi/2$ behind I.



Current phasor I, is represented here arbitrarily.

$V_{Rm} \rightarrow$ Maximum value of voltage across R.

$\vec{V} \rightarrow$ Resultant voltage

$$\boxed{V_m^2 = V_{Rm}^2 + (V_{cm} - V_{Lm})^2} \quad (3)$$

we know,

$$\left.\begin{aligned} V_{Rm} &= i_m R \\ V_{Lm} &= i_m X_L \\ V_{cm} &= i_m X_c \end{aligned}\right\} \rightarrow (4)$$

substituting (4) in (3),

$$v_m^2 = i_m^2 R^2 + (i_m X_C - i_m X_L)^2$$

$$v_m^2 = i_m^2 (R^2 + [X_C - X_L]^2)$$

$$v_m = i_m \left[R^2 + (X_C - X_L)^2\right]^{1/2}$$

$$i_m = \frac{v_m}{\sqrt{R^2 + (X_C - X_L)^2}} \quad \text{— (5)}$$

$$i_m = \frac{v_m}{Z} \quad \text{— (6)}$$

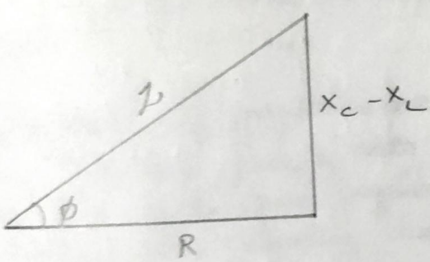
(The above eqn. gives the amplitude of the current)

where $Z = \sqrt{R^2 + (X_C - X_L)^2}$

Here 'Z' represents the impedance of the circuit. (Effective resistance offered by the circuit).

Sub. (5) in eqn. (2)

$$i = \frac{v_m \sin(\omega t + \phi)}{\sqrt{R^2 + (X_C - X_L)^2}}$$



Impedance diagram

$$\tan\phi = \frac{\text{opp.}}{\text{adj.}} = \frac{V_{Cm} - V_{Lm}}{V_{Rm}}$$

$$\tan\phi = \frac{i_m X_C - i_m X_L}{i_m R}$$

$$\tan\phi = \left(\frac{X_C - X_L}{R}\right)$$

$$\therefore \phi = \tan^{-1}\left(\frac{X_C - X_L}{R}\right)$$

If $X_C > X_L$, circuit is capacitive

If $X_C < X_L$, circuit is Inductive

Additional Information:

In the phasor diagram of this series LCR circuit, the length of the phasors represent the magnitudes of the voltages (or) voltage amplitude across inductor (V_L), capacitor (V_C) and resistor (V_R).

The magnitude of the voltage across inductor (V_L) and capacitor (V_C) depends on

$$V_L = I \cdot \omega L$$

$$\text{and } V_C = \frac{I}{\omega C}$$

$V_{Lm} = i_m X_L$ where $X_L = \omega L$

$V_{Cm} = i_m X_C$ where $X_C = \frac{1}{\omega C}$

The reason $V_L < V_C$ is the circuit is operating below resonance frequency making capacitive reactance dominant.

* For phasor diagram and voltage, current Vs ωt graph refer textbook pg. No. 188.

* In the graph, $\phi > 0$, so current leads voltage. This is a capacitive circuit behaviour.

* I is drawn ahead of V by an angle ϕ, indicating current leads voltage.

* In the graph, solid blue curve represents voltage and dashed curve represents current.

Resonance :

Resonance is a phenomenon common among systems that have a tendency to oscillate at a particular frequency, known as their natural frequency.

If such a system is driven by an energy source whose frequency is near this natural frequency, the amplitude of oscillation becomes large.

Under this condition, the current in the circuit becomes maximum.

Examples :

(i) Swing – A child's swing has a natural frequency. Pushing it at just the right rhythm (matching that frequency) makes the swing go higher.

(ii) Tuning fork – If you strike one tuning fork and place it near another fork of the same frequency, the second starts vibrating due to resonance.

(iii) Glass breaking by sound : A loud sound at the glass's natural frequency can make it vibrate violently and shatter.

(iv) Radio tuning : The tuning circuit resonates at the selected station's frequency, allowing only that signal to be amplified.

(v) Microwave oven : Microwaves are tuned to the natural frequency of water molecules to heat food efficiently.

In LCR circuit,

$$i_m = \frac{V_m}{Z} \quad \text{(eqn. 6)}$$

$$i_m = \frac{V_m}{\sqrt{R^2 + (X_C - X_L)^2}} \quad \text{(eqn)}$$

$$i_m(max) = \frac{V_m}{R} \quad -(7)$$

If $X_C = X_L$, impedance is minimum. $(Z = \sqrt{R^2 + 0^2} = R)$

$$\frac{1}{\omega C} = \omega L$$

upon cross multiplication

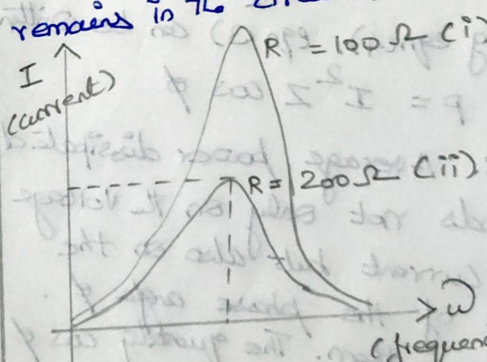
$$\omega^2 LC = 1$$

$$\omega^2 = \frac{1}{LC}$$

$$\therefore \omega_0 = \frac{1}{\sqrt{LC}} \quad \text{(Resonant frequency)}$$

(I – maximum)

$X_C = X_L$ says, they were out in phase (same, therefore cancels out) and only resistance R remains in the circuit.



Since the value of R is minimum in case (i) the current amplitude for case (i) is twice that for case (ii)

Power in AC circuit: The power factor)

Applied to a series LCR circuit

$V = V_m \sin \omega t$

$i = i_m \sin(\omega t + \phi)$

$p = vi = (V_m \sin \omega t)(i_m \sin \omega t + \phi)$

$p = V_m i_m \sin(\omega t) \sin(\omega t + \phi)$ (1)

Sub. (2) in (1)

$$p = \frac{V_m i_m}{2}\left[\cos\phi - \cos(2\omega t + \phi)\right]$$

(since the second term is time-dependent, it's average is zero.)

Average power over a cycle,

$$\bar{P} = \frac{V_m i_m}{2}\cos\phi$$

$$= \frac{V_m}{\sqrt{2}} \frac{i_m}{\sqrt{2}} \cos\phi$$

$P = VI\cos\phi \rightarrow (3)$

we know $i_m = \frac{V_m}{Z} \rightarrow (4)$

using eqn(4). eqn (3) can be written as $p = I^2 Z \cos\phi$

$\therefore$ Average power dissipated depends not only on the voltage and current but also on the cosine of the phase angle ϕ between them. The quantity $\cos\phi$ is called the power factor.

using the formula:

$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

Here $A = \omega t$ and $B = \omega t + \phi$

$\sin(\omega t)\sin(\omega t + \phi) = \frac{1}{2}\left[\cos(\omega t - \omega t - \phi) - \cos(\omega t + \omega t + \phi)\right]$

$= \frac{1}{2}\left[\cos(-\phi) - \cos(2\omega t + \phi)\right]$

$\cos(-\phi) = \cos\phi$.

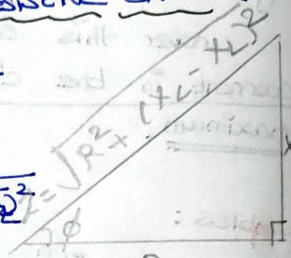
$\therefore \sin(\omega t)\sin(\omega t + \phi) = \frac{1}{2}\left[\cos\phi - \cos(2\omega t + \phi)\right] \rightarrow (2)$

If you understand this, you can substitute directly

case (i): Resistive circuit.

$\cos\phi = \frac{\text{base}}{\text{hyp.}}$

$\therefore \cos\phi = \frac{R}{\sqrt{R^2 + (X_C - X_L)^2}}$



$\cos\phi = \frac{R}{R} = 1.$

$\cos\phi = \cos 0$

$\therefore \boxed{\phi = 0.}$ V and I are in same phase. $\therefore$ Maximum power is dissipated. A resistive circuit dissipates maximum power. It is not storing power in the circuit.

case (ii): purely inductive (or) capacitive circuit:

In Inductor, current lags behind $\pi/2$ & in capacitor current leads voltage by $\pi/2$

$\therefore \phi = \pm\frac{\pi}{2}$

$\cos\phi = \cos\left(\pm\frac{\pi}{2}\right)$

$\cos\phi = 0$

$p = i^2 Z \cos\phi$

$p = 0.$

No power dissipation. $\therefore$ It indicates that inductor & capacitor stores energy. current through such circuits are called wattless current.