Uniformly Charged Sphere

Let us consider a sphere of radius r_0 having a uniform volume charge density of ρ_v C/m^3. To determine $\vec{D}$ everywhere, inside and outside the sphere, we construct Gaussian surfaces ofradius $r < r_0$ and $r > r_0$ as shown in Fig. 2.6 (a) and Fig. 2.6(b).

For the region $r \le r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r^3 \quad \text{.........................(2.18)}$$

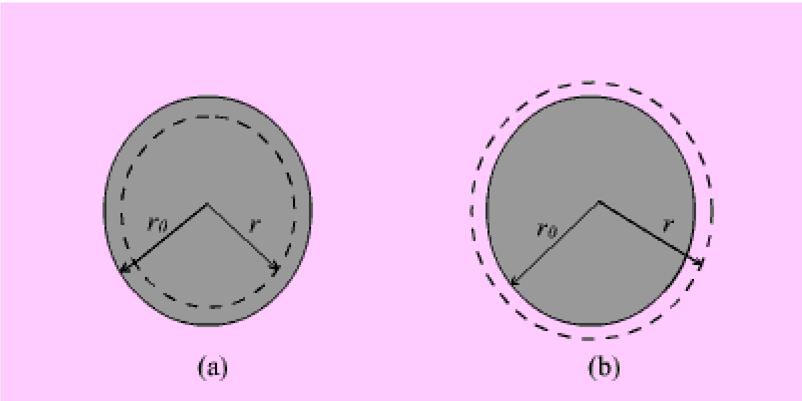


Fig : Uniformly Charged Sphere

By applying Gauss's theorem,

$$\oint_s \vec{D}\cdot d\vec{s} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} D_r r^2 \sin\theta d\theta d\phi = 4\pi r^2 D_r = Q_{en} \quad \text{...............(2.19)}$$

Therefore

$$\vec{D} = \frac{r}{3}\rho_v \hat{a}_r \qquad 0 \le r \le r_0 \quad \text{..(2.20)}$$

For the region $r \ge r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r_0^3 \quad \text{...(2.21)}$$

By applying Gauss's theorem,

$$\vec{D} = \frac{r_0^3}{3r^2}\rho_v \hat{a}_r \qquad r \geq r_0 \qquad \text{...(2.22)}$$

Divergence of a Vector Field:

In study of vector fields, directed line segments, also called flux lines or streamlines, represent field variations graphically. The intensity of the field is proportional to the density of lines. For example, the number of flux lines passing through a unit surface *S*normal to the vector measures the vector field strength.

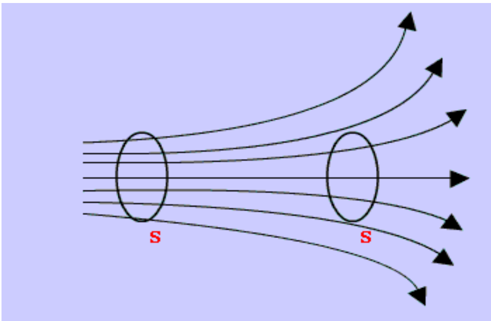


Fig : Flux Lines

We have already defined flux of a vector field as

$$\psi = \int_S A\cos\theta ds = \int_S \vec{A}\cdot\hat{a}_n ds = \int_S \vec{A}\cdot d\vec{s} \qquad \text{...(1.57)}$$

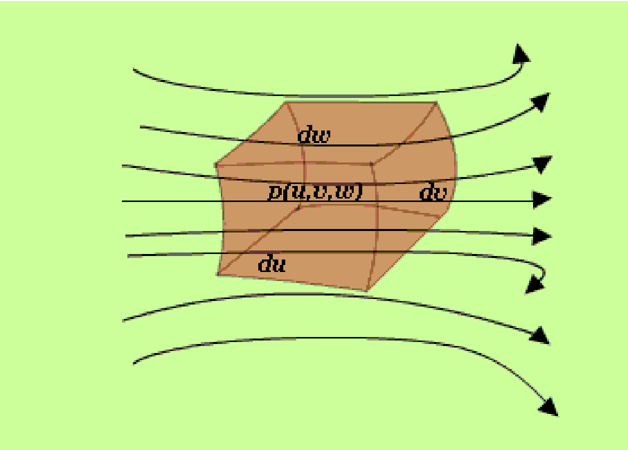
For a volume enclosed by a surface,

$$\psi = \oint_S \vec{A}\cdot d\vec{s} \qquad \text{...(1.58)}$$

We define the divergence of a vector field $\vec{A}$at a point *P* as the net outward flux from a volume enclosing *P*, as the volume shrinks to zero.

$$div\ \vec{A} = \nabla\cdot\vec{A} = \lim_{\Delta v \to 0}\frac{\oint_s \vec{A}\cdot d\vec{s}}{\Delta v} \qquad \text{...(1.59)}$$

Here $\Delta V$is the volume that encloses *P* and *S* is the corresponding closed surface.



Let us consider a differential volume centered on point P(u,v,w) in a vector field $\vec{A}$ The flux through an elementary area normal to u is given by ,

$$\phi_u = \vec{A}\cdot\hat{a}_u h_2 h_3 dv dw$$ ....(1.60)

Net outward flux along u can be calculated considering the two elementary surfaces perpendicular to u .

$$\left[h_2h_3A_u\Big|_{\left(u+\frac{du}{2},v,w\right)} - h_2h_3A_u\Big|_{\left(u-\frac{du}{2},v,w\right)} \right] dvdw \cong \frac{\partial\left(h_2h_3A_u\right)}{\partial u} dudvdw$$
....(1.61)

Considering the contribution from all six surfaces that enclose the volume, we can write

$$\operatorname{div}\vec{A} = \nabla\cdot\vec{A} = \lim_{\Delta v\to 0}\frac{\oint_s \vec{A}\cdot\vec{ds}}{\Delta v} = \frac{dudvdw\frac{\partial\left(h_2h_3A_u\right)}{\partial u} + dudvdw\frac{\partial\left(h_1h_3A_v\right)}{\partial v} + dudvdw\frac{\partial\left(h_1h_2A_w\right)}{\partial w}}{h_1h_2h_3dudvdw}$$

$$\therefore\ \nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial\left(h_2h_3A_u\right)}{\partial u} + \frac{\partial\left(h_1h_3A_v\right)}{\partial v} + \frac{\partial\left(h_1h_2A_w\right)}{\partial w}\right]$$
....(1.62)

Hence for the Cartesian, cylindrical and spherical polar coordinate system, the expressions for
divergence cawritten as:

In Cartesian coordinates:

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$ ....(1.63)

In cylindrical coordinates:

$$\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial\left(\rho A_\rho\right)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

....(1.64)

and in spherical polar coordinates:

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta A_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

....(1.65)

In connection with the divergence of a vector field, the following can be noted

- Divergence of a vector field gives a scalar.
- $\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$ ....(1.66)

  $\nabla\cdot(V\vec{A}) = V\nabla\cdot\vec{A} + \vec{A}\cdot\nabla\vec{V}$

Divergence theorem :

Divergence theorem states that the volume integral of the divergence of vector field isequal to the net outward flux of the vector through the closed surface that bounds the

volume. Mathematically, $\int_V \nabla\cdot\vec{A}dv = \oint_S \vec{A}\cdot d\vec{s}$

Proof:

Let us consider a volume V enclosed by a surface S . Let us subdivide the volume in large number of cells. Let the k^{th} cell has a volume ΔV_K and the corresponding surface is denoted by S_k. Interior to the volume, cells have common surfaces. Outward flux through these common surfaces from one cell becomes the inward flux for the neighboring cells. Therefore when the total flux from these cells are considered, we actually get the net outward flux through the surface surrounding the volume. Hence we can write:

$$\oint_S \vec{A}\cdot d\vec{s} = \sum_k \oint_{S_k} \vec{A}\cdot d\vec{s} = \sum_k \frac{\oint_{S_k} \vec{A}\cdot d\vec{s}}{\Delta V_k}\Delta V_k$$

....(1.67)

In the limit, that is when $K\to\infty$ and $\Delta V_K\to 0$ the right hand of the expression can bewritten as $\int_V \nabla.A\mathrm{d}V$

$$\oint_S \vec{A}\cdot d\vec{S} = \int_V \nabla\cdot A dV$$

Hence we get , which is the divergence theorem