TOPIC : 1.3 – Discrete Random Variables

Random variable

A random variable is a rule that assigns a numerical values to each possible outcome of an experiment.

A real-valued function defined on the outcome of a probability experiment is called a random variable.

1. Discrete Random Variable
2. Continuous Random Variable

Discrete Random Variable

A Discrete random variable is a random variable whose possible values constitute finite set of values or countably infinite set of values.

Probability Mass Function (PMF)

Let X be a one dimensional discrete random variable which takes the values $x_1, x_2, \cdots x_n$. To each possible outcome 'x_i', we can associate a number p_i, i.e. $P[X = x_i] = P_i$, called the probability of x_i.

The numbers P_i satisfies the following conditions (i) $p(x_i) \geq 0 \quad \forall i$

(ii) $\sum_{i=1}^{\infty} p(x_i) = 1$.

The function $p(x)$ satisfying the above two conditions is called the Probability Mass Function.

Cumulative distribution (or) Distribution function of X

The cumulative distribution function $F(x)$ of a discrete random variable X with probability distribution $P(x)$ is given by

$$F(x) = P(X \le x) = \sum_{t \le x} p(t)$$

$$x = -\infty, \ldots, -1, 0, 1, \ldots \infty.$$

Properties of distribution function

(i) $F(-\infty) = P(X \le -\infty) = 0$

(ii) $F(\infty) = P(X \le \infty) = 1$

(iii) $F(x_1) \le F(x_2)$ if $x_1 \le x_2$

(iv) $P(X > x) = 1 - P(X \le x)$

(v) $P(X \le x) = 1 - P(X > x)$

Expected value of a Discrete random variable X

Let X be a discrete random variable assuming values $x_1, x_2, \dots x_n$ with corresponding probabilities $P_1, P_2, \dots P_n$. Then

$$E[X] = \sum_{i=1}^{n} x_i \, p(x_i)$$

is called the Expected value of X (or) Mean of X

The Variance of a Discrete random variable X

It is defined by $\text{Var}(X) = E[X^2] - [E(X)]^2$

① Find the expected value of the discrete random variable X with the pmf $p(x) = \begin{cases} 1/3, & x = 0 \\ 2/3, & x = 2 \end{cases}$

$$E(X) = \sum x \, p(x)$$

$$= (0)(1/3) + (2)(2/3)$$

$$= 4/3$$

③ A random variable X has the following probability function

x	0	1	2	3	4	5	6	7	8
$P(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Determine the value of 'a'

(ii) Find $P(x<3)$, $P(x \geq 3)$, $P(0<x<5)$

(iii) Find the distribution function of X.

(i) WKT $\sum p(x) = 1$

$$\Rightarrow a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$$

$$\Rightarrow 81a = 1$$

$$\Rightarrow \boxed{a = \frac{1}{81}}$$

(ii) $P(x<3) = P(x=0) + P(x=1) + P(x=2)$

$$= a + 3a + 5a = 9a = \frac{9}{81}$$

$P(x \geq 3) = 1 - P(x<3)$

$$= 1 - \frac{9}{81} = \frac{72}{81}$$

$P(0<x<5) = P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$$= 3a + 5a + 7a + 9a = 24a = \frac{24}{81}$$

(iii) To find the distribution function of x.

$X = x$	$F(x) = P(X \le x)$
0	$F(0) = P(X \le 0) = \frac{1}{81}$
1	$F(1) = P(X \le 1) = 4a = \frac{4}{81}$
2	$F(2) = P(X \le 2) = 9a = \frac{9}{81}$
3	$F(3) = P(X \le 3) = 16a = \frac{16}{81}$
4	$F(4) = P(X \le 4) = 25a = \frac{25}{81}$
5	$F(5) = P(X \le 5) = \frac{36}{81}$
6	$F(6) = P(X \le 6) = \frac{49}{81}$
7	$F(7) = P(X \le 7) = 64a = \frac{64}{81}$
8	$F(8) = P(X \le 8) = 81a = 1$

5. A random variable X has the following probability distribution

X	0	1	2	3	4	5	6	7
P(X)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

Find (i) the value of K (ii) $P[1.5 < x < 4.5 / x > 2]$ (iii) the smallest value of n for which $P[x \le n] > \frac{1}{2}$

(i) WKT $\sum p(x) = 1$

$\Rightarrow \quad K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$

$\Rightarrow \quad 9K + 10K^2 - 1 = 0$

$\Rightarrow \quad K = \frac{-9 \pm \sqrt{81 + 40}}{20} = \frac{-9 \pm \sqrt{121}}{20}$

$= \frac{-9+11}{20}, \frac{-9-11}{20} = \frac{1}{10}, -1$

$\boxed{K = \frac{1}{10}}$

(ii) $P[1.5 < x < 4.5 / x > 2]$

$= \frac{P[(1.5 < x < 4.5) \cap (x > 2)]}{P[x > 2]}$

$= \frac{P[2 < x < 4.5]}{P[x > 2]} \rightarrow (1)$

$P[2 < x < 4.5] = P[x = 3] + P[x = 4]$

$= 2K + 3K = \frac{5}{10} = \frac{1}{2}$

$P[x > 2] = 1 - P[x \le 2]$

$= 1 - \{K + 2K\} = 1 - 3K = 1 - \frac{3}{10} = \frac{7}{10}$

Sub. in (1)

$P[1.5 < x < 4.5 / x > 2] = \frac{5/10}{7/10} = \frac{5}{7}$

(iii)

$X = x$	$F(x) = P(X \le x)$
0	$F(0) = P(X \le 0) = 0$
1	$F(1) = P(X \le 1) = k = 1/10$
2	$F(2) = P(X \le 2) = 3k = \frac{3}{10}$
3	$F(3) = P(X \le 3) = 5k = \frac{1}{2}$
4	$F(4) = P(X \le 4) = 8k = \frac{4}{5}$
5	$F(5) = P(X \le 5) = 8k + k^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$
6	$F(6) = P(X \le 6) = 8k + 3k^2 = \frac{8}{10} + \frac{3}{100} = \frac{83}{100}$
7	$F(7) = P(X \le 7) = 1$

The smallest value of n for which

$P[X \le n] > 1/2$ is $\boxed{n = 4}$

9) Given the following probability distribution of x compute (i) $E[x]$ (ii) $E[x^2]$ (iii) $E[2x \pm 3]$ (iv) $Var[2x \pm 3]$

x	-3	-2	-1	0	1	2	3
$p(x)$	0.05	0.1	0.3	0	0.3	0.15	0.1

(i) $E[x] = \sum x\, p(x)$

$= (-3)(0.05) + (-2)(0.1) + (-1)(0.3) + (1)(0.3) + 2(0.15) + 3(0.1)$

$= 0.25$

(ii) $E(x^2) = \sum x^2 p(x)$

$= (9)(0.05) + (4)(0.1) + (1)(0.3) + (1)(0.3) + (4)(0.15) + (9)(0.1)$

$= 2.95$

(iii) $E[2x \pm 3] = 2E[x] \pm 3$

$= 2(0.25) \pm 3$

$= 0.5 \pm 3$

(iv) $Var(x) = E[x^2] - [E(x)]^2$

$= 2.95 - (0.25)^2$

$= 2.95 - 0.0625$

$= 2.8875$

$Var(2x \pm 3) = 4\, Var(x)$

$= 11.55$