TOPIC : 1.5 – Continuous Random Variables

Continuous Random Variable

A random variable X which takes all possible values in a given interval is called a continuous Random variable.

[Prob]ability density function is a function such [t]hat

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(iii) $P(a \leq X \leq b) = \int_a^b f(x)\,dx$

Cumulative Distribution Function

If $f(x)$ is a pdf of a continuous random variable X, then the function

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx, \quad -\infty < x < \infty$$

is called the cumulative distribution function of the random variable X.

Formula

(i) $f(x) = \frac{d}{dx}[F(x)]$

(ii) Mean $= E[X] = \int_{-\infty}^{\infty} x\,f(x)\,dx$

(iii) $E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

(iv) $P[a \leq X \leq b] = F(b) - F(a)$

(v) $P[a \le X \le b] = P[a \le X < b] = P[a < X \le b]$
$= P[a < X < b]$, X being a continuous random variable.

① If the pdf of a random variable X is $f(x) = \frac{x}{2}$ in $0 \le x \le 2$, find $P[X > 1.5 / X > 1]$.

$$P[X > 1.5 / X > 1] = \frac{P[(X > 1.5) \cap (X > 1)]}{P[X > 1]}$$

$$= \frac{P[X > 1.5]}{P[X > 1]} \rightarrow ①$$

$$P[X > 1.5] = \int_{1.5}^{2} f(x)\,dx = \int_{1.5}^{2} \frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_{1.5}^{2} = \frac{1}{4}[4 - 2.25]$$

$$= \frac{1}{4}[1.75] = 0.4375$$

$$P[X > 1] = \int_{1}^{2} f(x)\,dx = \int_{1}^{2} \frac{x}{2}\,dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_{1}^{2}$$

$$= \frac{1}{4}[4 - 1] = 0.75$$

sub in ①,

$$P[X > 1.5 / X > 1] = \frac{0.4375}{0.75} = 0.5833$$

2) Show that the function $f(x) = e^{-x}, x \geq 0$ is a probability density function of a random variable of x

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_0^{\infty}$$

$$= -\left[0-1\right] = 1$$

$\therefore$ Given $f(x)$ is a pdf.

3) If $f(x) = \begin{cases} Ke^{-x}, & x>0 \\ 0, & \text{elsewhere} \end{cases}$ is the pdf of a random variable X, then find the value of K.

WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{0}^{\infty} Ke^{-x}\,dx = 1 \Rightarrow K\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

$$\Rightarrow -K\left[0-1\right] = 1 \Rightarrow \boxed{K=1}$$

4) Assume that X is a continuous random variable with pdf $f(x) = \begin{cases} \frac{3}{4}(2x-x^2), & 0 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$ Find $P[X>1]$

$$P[X>1] = \int_1^2 f(x)\,dx = \int_1^2 \frac{3}{4}(2x-x^2)\,dx$$

$$= \frac{3}{4}\left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right]_1^2 = \frac{3}{4}\left[\left(4-\frac{8}{3}\right)-\left(1-\frac{1}{3}\right)\right]$$

$$= \frac{3}{4}\left[\frac{4}{3}-\frac{2}{3}\right] = \frac{3}{4}\left[\frac{2}{3}\right] = \frac{1}{2}$$

(15) The pdf of a random variable X is given by $f(x) = \begin{cases} x, & 0 < x < 1 \\ k(2-x), & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$. Find

(i) the value of k

(ii) $P[0.2 < X < 1.2]$

(iii) $P[0.5 < X < 1.5 / X \geq 1]$

(iv) the distribution function of X.

(i) WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int^{1} \ldots \int^{2} k(2-x)\,dx = 1$

$\Rightarrow \frac{1}{2} + k\left[2 - \frac{3}{2}\right] = 1$

$\Rightarrow k\left[\frac{1}{2}\right] = \frac{1}{2} \Rightarrow \boxed{k = 1}$

(ii) $P[0.2 < X < 1.2] = \int_{0.2}^{1.2} f(x)\,dx$

$= \int_{0.2}^{1} x\,dx + \int_{1}^{1.2} (2-x)\,dx$

$= \left(\frac{x^2}{2}\right)_{0.2}^{1} + \left[2x - \frac{x^2}{2}\right]_{1}^{1.2}$

$= \frac{1}{2}\left[1 - 0.04\right] + \left[\left(2.4 - 0.72\right) - \left(2 - \frac{1}{2}\right)\right]$

$= \frac{1}{2}\left[0.96\right] + \left[0.18\right] = 0.66$

(iii) $P[0.5 < X < 1.5 / X \geq 1]$

$= \frac{P[1 < X < 1.5]}{P[X \geq 1]} \rightarrow (1)$

$P[1 < X < 1.5] = \int_{1}^{1.5} (2-x)\,dx = \left[2x - \frac{x^2}{2}\right]_{1}^{1.5}$

$= \left[\left(3 - \frac{2.25}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$

$= \left[3 - 1.125 - 1.5\right] = 0.375$

$P[X \geq 1] = \int_{1}^{2} (2-x)\,dx = \left[2x - \frac{x^2}{2}\right]_{1}^{2} = \left[(4-2) - \left(2 - \frac{1}{2}\right)\right] = 0.5$

Sub. in ①

$$P[0.5 < X < 1.5 / X \geq 1] = \frac{0.375}{0.5} = 0.75$$

(iv) To find CDF

<u>when $0 < x < 1$</u>

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{x} x\,dx = \left(\frac{x^2}{2}\right)_0^x$$

$$= \frac{x^2}{2}$$

<u>when $1 < x < 2$</u>

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{1} x\,dx + \int_{1}^{x} (2-x)\,dx$$

$$= \left(\frac{x^2}{2}\right)_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x$$

$$= \frac{1}{2} + \left[\left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} - \frac{3}{2} + 2x - \frac{x^2}{2}$$

$$= 2x - \frac{x^2}{2} - 1$$

<u>when $x > 2$</u>

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{1} x\,dx + \int_{1}^{2} (2-x)\,dx + \int_{2}^{x} 0\,dx$$

$$= \left(\frac{x^2}{2}\right)_0^1 + \left(2x - \frac{x^2}{2}\right)_1^2 = \frac{1}{2} + \left(4 - \frac{1}{2}2\right) - \left(2 - \frac{1}{2}\right)$$

$$= \frac{1}{2} + 2 - \frac{3}{2} = 1$$

$$\therefore F(x) = \begin{cases} 0 & , x < 0 \\ x^2/2 & , 0 < x < 1 \\ 2x - \frac{x^2}{2} - 1 & , 1 < x < 2 \\ 1 & , x > 2 \end{cases}$$

(11) If the density function of a continuous r.v X is given by $f(x) = \begin{cases} ax & , 0 \le x \le 1 \\ a & , 1 \le x \le 2 \\ 3a - ax & , 2 \le x \le 3 \\ 0 & , \text{otherwise} \end{cases}$

find (i) the value of 'a'
(ii) cdf of X

(i) WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$

$$a\left(\frac{x^2}{2}\right)_0^1 + a\left(x\right)_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1$$

$$\Rightarrow a\left(\frac{1}{2}\right) + a + \left[\left(9a - \frac{9a}{2}\right) - \left(6a - 2a\right)\right] = 1$$

$$\Rightarrow \frac{3a}{2} + \left[5a - \frac{9a}{2}\right] = 1 \Rightarrow \frac{3a}{2} + \frac{a}{2} = 1$$

$$\Rightarrow 2a = 1 \Rightarrow \boxed{a = 1/2}$$

(ii) <u>To find CDF</u>

<u>when $0 \le x \le 1$</u> $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^x \frac{x}{2}\,dx$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^x = \frac{x^2}{4}.$$

when $1 \le x \le 2$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^1 \frac{x}{2}\,dx + \int_1^x \frac{1}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^x$$

$$= \frac{1}{4} + \frac{1}{2}(x-1) = \frac{1}{2}\left(x - \frac{1}{2}\right)$$

when $2 \le x \le 3$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{2}\,dx + \int_2^x \left(\frac{3}{2} - \frac{x}{2}\right)dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^2 + \left[\frac{3}{2}x - \frac{x^2}{4}\right]_2^x$$

$$= \frac{1}{4} + \frac{1}{2} + \left[\left(\frac{3}{2}x - \frac{x^2}{4}\right) - (3-1)\right]$$

$$= \frac{1}{4} + \frac{1}{2} - 2 + \frac{3}{2}x - \frac{x^2}{4}$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{4}, & 0 \le x \le 1 \\ \frac{1}{2}(x - \frac{1}{2}), & 1 \le x \le 2 \\ \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}, & 2 \le x \le 3 \\ 1, & x > 3 \end{cases}$$