



TOPIC : 1.10 – Exponential Distribution

Exponential Distribution

A continuous random variable X is said to follow an exponential distribution with parameter $\lambda > 0$, if its pdf is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

1. Find the moment generating function of the exponential distribution and hence find mean and variance.

The pdf of the exponential distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\text{MGF } M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$\begin{aligned} &= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{-x(\lambda-t)} dx \\ &= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^{\infty} = \frac{\lambda}{t-\lambda} [0-1] = \frac{\lambda}{\lambda-t} \end{aligned}$$

$$\begin{aligned} \text{Mean} &= \left[\frac{d}{dt} \{ M_x(t) \} \right]_{t=0} = \left[\frac{d}{dt} \left\{ \frac{\lambda}{\lambda-t} \right\} \right]_{t=0} \\ &= \lambda \left[\frac{-1}{(\lambda-t)^2} (-1) \right]_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda} \end{aligned}$$

$$\begin{aligned} E[x^2] &= \left[\frac{d^2}{dt^2} \{ M_x(t) \} \right]_{t=0} = \left[\frac{d}{dt} \left\{ \frac{\lambda}{(\lambda-t)^2} \right\} \right]_{t=0} \\ &= \lambda \left[\frac{-2}{(\lambda-t)^3} (-1) \right]_{t=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2} \end{aligned}$$

$$\text{Var}(x) = E[x^2] - [E(x)]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$



State and prove memoryless property for exponential distribution.

If the random variable X follows exponential distribution, then

$P[X > s+t | X > t] = P[X > s]$ for all $s, t > 0$
called the memoryless property.

∴

The pdf of X $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$.

$$P[X > s+t | X > t] = \frac{P[X > s+t \cap X > t]}{P[X > t]}$$

$$= \frac{P[X > s+t]}{P[X > t]} \rightarrow (1)$$

$$P[X > t] = \int_t^{\infty} f(x) dx = \int_t^{\infty} \lambda e^{-\lambda x} dx$$
$$= \lambda \left(\frac{e^{-\lambda x}}{-\lambda} \right)_t^{\infty} = - [0 - e^{-\lambda t}] = e^{-\lambda t}$$

$$P[X > s+t] = e^{-\lambda(s+t)} \quad \dots \quad (2)$$

sub. in (1),

$$P[X > s+t | X > t] = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = \frac{e^{-\lambda s} e^{-\lambda t}}{e^{-\lambda t}}$$
$$= e^{-\lambda s}$$
$$= P[X > s]$$



The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$.
What is the prob. that a repair takes (i) at least 10 h
Given (ii) the repair

Let x denote the repair time

Given x is exponential with $\lambda = \frac{1}{2}$

$$\begin{aligned} \therefore \text{pdf of } x \quad f(x) &= \lambda e^{-\lambda x}, \quad x \geq 0 \\ &= \frac{1}{2} e^{-x/2}, \quad x \geq 0 \end{aligned}$$

$$\begin{aligned} \therefore P[x > 10 / x > 9] &= P[x > 9+1 / x > 9] \\ &= P[x > 1] \quad \text{by memoryless property} \\ &= e^{-\lambda(1)} = e^{-1/2} \quad [P(x > t) = e^{-\lambda t}] \\ &= 0.6065 \\ \text{(ii) } P[x > 2] &= e^{-\lambda(2)} = e^{-1/2(2)} = e^{-1} = 0.3679 \end{aligned}$$



④ If a continuous random variable X follows distribution in the interval $(0, 2)$ & continuous random variable Y follows exponential distribution with parameter α , find α such that $P[X < 1] = P[Y < 1]$.

Given X is uniformly distributed over $(0, 2)$

$$\therefore \text{pdf of } X \quad f(x) = \frac{1}{b-a} = \frac{1}{2}, \quad 0 < x < 2$$

and Y is exponentially distributed with parameter

$$\therefore \text{pdf of } Y \quad f(y) = \alpha e^{-\alpha y}, \quad y \geq 0.$$

$$\text{Given that } P(X < 1) = P(Y < 1)$$

$$\Rightarrow \int_0^1 f(x) dx = \int_0^1 \alpha e^{-\alpha y} dy$$

$$\Rightarrow \frac{1}{2} (x)_0^1 = \alpha \left(\frac{e^{-\alpha y}}{-\alpha} \right)_0^1$$

$$\Rightarrow \frac{1}{2} = - (e^{-\alpha} - 1) \Rightarrow e^{-\alpha} - 1 = -\frac{1}{2}$$

$$\Rightarrow e^{-\alpha} = \frac{1}{2}$$

$$\Rightarrow e^{\alpha} = 2 \quad \Rightarrow \boxed{\alpha = \log_e 2}$$