Normal distribution :-

A continuous random variable x is said to follow normal distribution, if is probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

$$-\infty < \mu < \infty, \quad \sigma > 0$$

with parameters mean (μ) and standard deviation (σ)

Note :-

Symbolically $x \sim N(\mu, \sigma^2)$ (or) $N(\mu, \sigma)$

Standard normal distribution :-

* The normal distribution $N(0,1)$ (μ, σ) is called standard normal distribution, whose density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-1/2 x^2}, \quad -\infty < x < \infty$$

Derive M.G.F of normal distribution and hence find its mean and variance.

Pdf of normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

To find MGF

$$M_x(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Put $z = \frac{x-\mu}{\sigma}$ | $x = \infty \Rightarrow z = \infty$

$\sigma z = x - \mu$ | $x = -\infty \Rightarrow z = -\infty$

$\sigma dz = dx$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z + \mu)} e^{-1/2 z^2} \cancel{\sigma}\, dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\sigma t z} e^{-1/2 z^2} dz.$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2[z^2 - 2\sigma t z]} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2[z^2 - 2\sigma t z + \sigma^2 t^2 - \sigma^2 t^2]} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z - \sigma t)^2} e^{1/2 \sigma^2 t^2} dz$$

$$= \frac{e^{\mu t + 1/2 \sigma^2 t^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2 (z - \sigma t)^2} dz .$$

$$= \frac{e^{\mu t + 1/2 \sigma^2 t^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du$$

$u = z - \sigma t$	$z = \infty \Rightarrow u = \infty$
$du = - dz$	$z = -\infty \Rightarrow u = -\infty$

$$= \frac{e^{\mu t + \frac{t^2 \sigma^2}{2}}}{\cancel{\sqrt{2\pi}}} \times \cancel{\sqrt{2\pi}} \qquad \because \int_{-\infty}^{\infty} e^{-u^2/2} = \sqrt{2\pi}$$

$$\boxed{M_x(t) = e^{\mu t + 1/2 \sigma^2 t^2}} \text{ ——— } (1)$$

To find mean and variance.

$(1) \Rightarrow M_x'(t) = e^{\mu t + 1/2 \sigma^2 t^2} (\mu + \sigma^2 t)$ ——— (2)

$(2) \Rightarrow M_x''(t) = e^{\mu t + 1/2 \sigma^2 t^2} \cdot (\sigma^2) + (\mu + \sigma^2 t)^2 e^{\mu t + 1/2 \sigma^2 t^2}$ ——— (3)

$(2) \Rightarrow$ Mean $= \mu_1' = M_x'(0) = \mu$

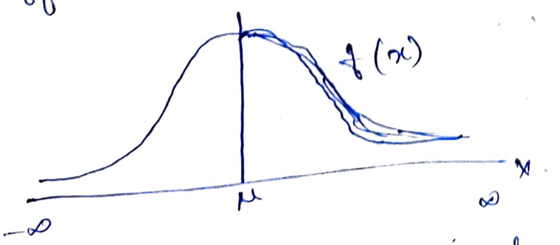
$(3) \Rightarrow \mu_2' = \sigma^2 + \mu^2$

variance $= \mu_2' - \mu_1'^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$

Hence $\boxed{\text{Mean} = \mu}$ & $\boxed{\text{Variance} = \sigma^2}$

Properties of normal distribution:-

* The pdf of normal distribution $N(\mu, \sigma^2)$ is shown in the figure



The normal curve is symmetrical about mean & bell shaped.

* The normal curve is a single peaked curve.
* The normal curve is asymptotic to x-axis (for)
* Mean, Median and mode coincide

* Mean of the normal distribution lies at the centre of normal curve.

① If X is a normal variant with mean 30 & standard deviation 5 find the following:

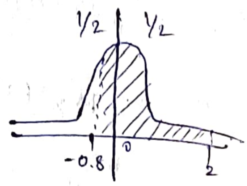
i). $P[26 \le x \le 40]$ ii). $P[X \ge 45]$ & iii). $P[|x - 30| > 5]$

Soln:

$X \sim N(30, 5^2)$

Let $Z = \frac{X-\mu}{\sigma}$ be the standard normal variant

total Prob = 1



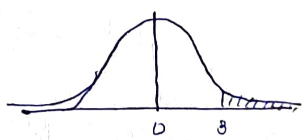
i). $P[26 \le x \le 40] = P\left[\frac{26-30}{5} \le \frac{X-\mu}{\sigma} \le \frac{40-30}{\sigma}\right]$

$= P[-0.8 \le z \le 2]$

$= P[0 \le z \le 0.8] + P[0 \le z \le 2]$

$= 0.2881 + 0.4772$

$= 0.7653$ //.



ii). $P[X \ge 45] = P\left[\frac{X-\mu}{\sigma} \ge \frac{45-30}{5}\right]$

$= P[z \ge 3]$

$= 0.5 - P[0 \le z \le 3]$

$= 0.5 - 0.4987$

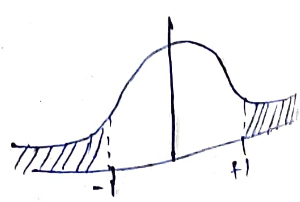
$= 0.0013$ //

iii). $P[|X-30| > 5] = 1 - [P(|-30| \le 5)]$

$= 1 - P[-5 \le x - 30 \le 5]$

$= 1 - P\left[\frac{-5}{5} \le \frac{x-30}{5} \le 5/5\right]$

$= 1 - P[-1 \le z \le 1]$



$= 1 - 2[0 \le z \le 1]$

$= 1 - 2(0.3413)$

$= 0.3174$ //.

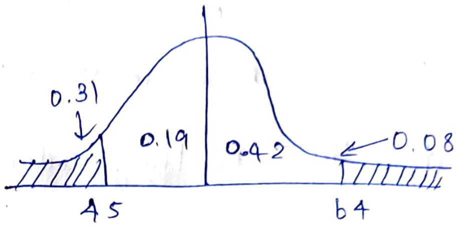
Q. In normal distribution 31% of items are under 45 and 8% are over 64. find the mean & standard deviation of the given distribution.

Soln:

Let m be the mean and σ be the standard deviation

Given: $P[X \le 45] = 0.31$ —— (1)

$P[X \ge 64] = 0.08$ —— (2)



(1) => $P[X \ge 64]$

(1) => $P[45 \le X \le m] = 0.19$

$P\left[\frac{45-m}{\sigma} \le \frac{X-\mu}{\sigma} \le 0\right] = 0.19$

$P\left[\frac{45-m}{\sigma} \le Z \le 0\right] = 0.19$

$P\left[0 \le Z \le \frac{m-45}{\sigma}\right] = 0.19$ —— (3)

from the table

$P[0 \le Z \le 0.5] = 0.19$ —— (4)

(2) => $P[m \le X \le 64] = 0.42$

$P\left[0 \le \frac{X-m}{\sigma} \le \frac{64-m}{\sigma}\right] = 0.42$

$P\left[0 \le Z \le \frac{64-m}{\sigma}\right] = 0.42$ —— (5)

From the table

$P[0 \le Z \le 1.41] = 0.42$ —— (6)

=> From (3) & (4)

$\frac{m-45}{\sigma} = 0.5$

from (5) and (6)

$\frac{64-m}{\sigma} = 1.41$

=> $m - 45 = 0.5\sigma$ —— (7)

$64 - m = 1.41\sigma$ —— (8)

(9)-(8) $19 = 1.91\sigma$

$\sigma = \frac{19}{1.9} \frac{\times 10}{\times 10} = \frac{190}{19} = 10$

$\sigma \approx 10$

(9) => $m - 45 = 0.5\sigma$

$m = 45 + 0.5(10)$

$= 45 + 5$

$\boxed{m = 50}$

Hence $\boxed{\text{Mean} = 50 \quad \text{Standard deviation} = 10}$