CAPACITANCE:

ELECTROSTATIC ENERGY and CAPACITANCE

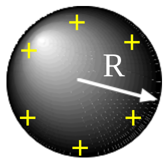
- Capacitance and capacitors
  - *Storage of electrical energy*
- Energy density of an electric field
- Combinations of capacitors
  - *In parallel*
  - *In series*

In the previous chapter, we saw that an object with charge Q, will have a

- Dielectrics
  - *Effects of dielectrics*
- Examples of capacitors

potential V. Conversely, if an object has a potential V it will have a charge Q. The

capacitance (C) of the object is the ratio Q/V.



Example: A charged spherical conductor with a charge Q. The potential of the sphere is

$$V = k\frac{Q}{R}.$$

Therefore, the *capacitance* of the charged sphere is

$$C = \frac{Q}{V} = \frac{Q}{k\,Q/R} = \frac{R}{k} = 4\pi\varepsilon_0 R$$

UNITS: Capacitance $\Rightarrow$ Coulombs/Volts

$\Rightarrow$ Farad (F).

Example: A sphere with R = 5.0 cm (= 0.05m)

$$C = 5.55 \times 10^{-12}\,\text{F}\ (\Rightarrow 5.55\text{pF}).$$

Capacitance is a measure of the "capacity" that an object has for "holding" charge, i.e., given two objects at the same potential, the one with the *greater capacitance* will have *more charge*. As we have seen, a charged object has potential energy (U); a device that is specifically designed to hold or store charge is called a *capacitor*.

Question 24.1: The Earth is a conductor of radius 6400 km. If it were an isolated sphere what would be its capacitance?

From before, the capacitance of a sphere is

$$C = 4\pi\varepsilon_0 R = 4\pi \times 8.85 \times 10^{-12} \times 6400 \times 10^{3}$$
$$= 7.1 \times 10^{-4}\,\mathrm{F}.$$

Earlier, question 22.4, we found the charge on the Earth was

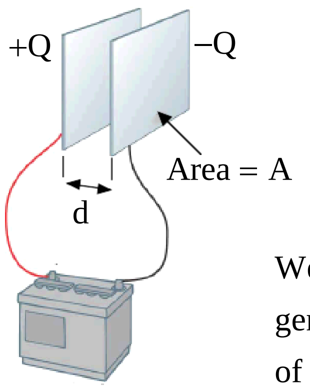
$$Q = -9.11 \times 10^{5}\,\mathrm{C}.$$

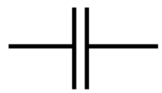
So, what is the corresponding potential? By definition

$$|V| = \frac{|Q|}{C} = \frac{9.11 \times 10^{5}}{7.1 \times 10^{-4}} = 1.28 \times 10^{9}\,\mathrm{V}$$

which is what we found in chapter 23 (question 23.7).

Capacitance of two parallel plates:





We can use a battery or a generator to move an amount of charge Q from one plate to the other. The electric field between the plates is:

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}. \qquad \text{(\textit{From ch. 22})}$$

Also, the potential difference (*voltage*) between the two plates is:

$$V = E.d = \frac{\sigma}{\varepsilon_0} d. \qquad \text{(\textit{From ch. 23})}$$

So the capacitance of this pair of plates is

$$C = \frac{Q}{V} = \frac{\sigma A}{V} = \varepsilon_0 \frac{A}{d}$$

Two parallel plates (Practical considerations):

Example: $A = 5.0\text{ cm} \times 5.0\text{ cm}$ with $d = 0.5$ cm.

$$C = \varepsilon_0 \frac{A}{d} = 4.4 \times 10^{-12}\text{F} = 4.4\text{pF}.$$

The maximum possible value of E in air (from earlier) $\approx 3 \times 10^6$ V/m. Therefore, the maximum potential difference (voltage) we can get between this pair of plates (in air) is:

$$V_{max} = E_{max}.d \approx 3 \times 10^6 \times 0.005 \approx 15{,}000\text{ V}.$$

Note: V_{max} *depends only on the spacing*.

Also, the maximum charge we can achieve is

$$Q_{max} = CV_{max} = 4.4 \times 10^{-12} \times 15 \times 10^3 = 66\text{ nC}.$$

A pair of parallel plates is a useful capacitor. Later, we will find an expression for the amount of energy stored.

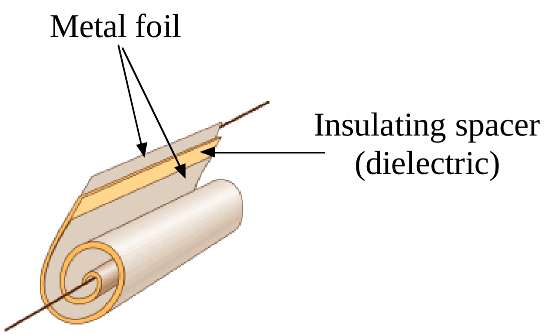
To have a 1F capacitance the area would have to be ~ 5.6×10^8 m^2, i.e., the length of each side of the plates would be ~ 23.8 km (i.e., about 14 miles!) with a spacing of 0.50 cm.

Two parallel plates:

$$C = \varepsilon_0 \frac{A}{d}$$

How can we increase the capacitance, i.e, get more charge per unit of potential difference?

- *increase* A
- *decrease* d



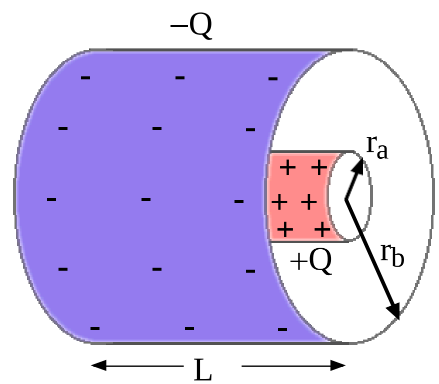
The capacitor is rolled up into a cylindrical shape.

- *increase* ε_0

by changing the medium between the plates, i.e.,

$\varepsilon_0 \Rightarrow \varepsilon = \kappa\varepsilon_0$ (later).

A cylindrical (coaxial) capacitor:

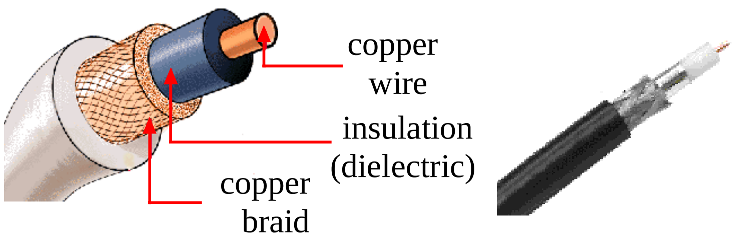


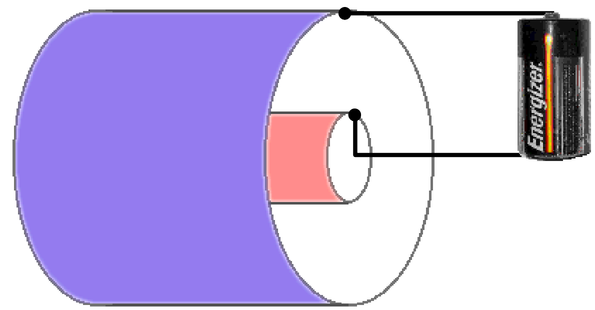
The capacitance of an air-filled coaxial capacitor of length L is:

$$C = \frac{2\pi\varepsilon_0 L}{\ln\left(r_b/r_a\right)}$$

A cylindrical (coaxial) capacitor (continued):

Example: coaxial (antenna) wire.





Assume an outer conductor (braid) radius $r_b \approx 2.5$mm, and an inner conductor (wire) radius $r_a \approx 0.5$mm, with neoprene insulation (dielectric) ($\varepsilon_0 \Rightarrow \kappa\varepsilon_0 = 6.9\varepsilon_0$).
The capacitance per meter is:

$$\frac{C}{L} = \frac{2\pi\kappa\varepsilon_0}{\ln\left(r_b / r_a\right)}$$

$$= \frac{2\times\pi\times 6.9\times 8.85\times 10^{-12}}{1.609} = 2.384\times 10^{-10}\ \text{F/m}$$

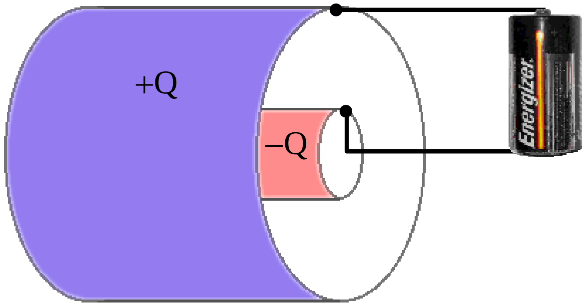
$$= 238.4\ \text{pF/m}$$

Question 24.2: What is the relationship between the *charge density* on the inner and outer plates of a cylindrical capacitor? Is it

A: larger on the outer plate?

B: larger on the inner plate?

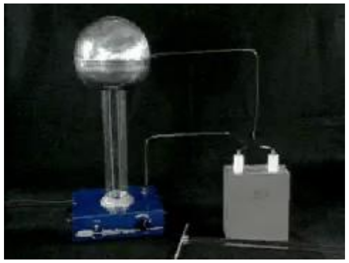
C: the same on both plates.



When an uncharged capacitor is connected to a source of potential difference (like a battery), charges move from one plate to the other. Therefore, the *magnitude* of the charge (±Q) on each plate is always the same. But the charge density depends on area ($\sigma = Q/A$); because the inner plate has a smaller area than the outer plate, the charge density on the inner plate is greater than the charge density on the outer plate.

Therefore, the answer is B (larger on the inner plate).

Storing energy in a capacitor



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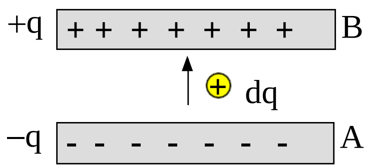
Because work is done to move charges onto the plates of a capacitor, the capacitor stores energy, *electrostatic potential energy*. The energy is released when the capacitor is discharged.

Where is the energy stored?

... in the electric field (between the plates), which has been produced during the charging process!

Storing energy in a parallel plate capacitor:



To store energy in a capacitor we "charge" it, producing an electric field between the plates. We do work moving charges from plate A to plate B. If the plates already have charge $\pm q$ and dq is then moved from A to B, the incremental work done is

$$dW = -dq(V_A - V_B),$$

where V_A and V_B are the potentials of plates A and B, respectively. This, then is the incremental *increase* in potential energy, dU, of the capacitor system.

If $V = V_B - V_A\ (V_B > V_A)$, then $dU = Vdq$.

But, by definition: $V = \frac{q}{C}$

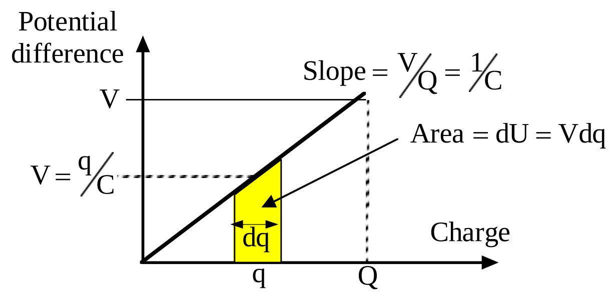
so the incremental increase in energy when dq of charge is taken from $A \rightarrow B$ is:

$$\therefore dU = \left(\frac{q}{C}\right)dq.$$

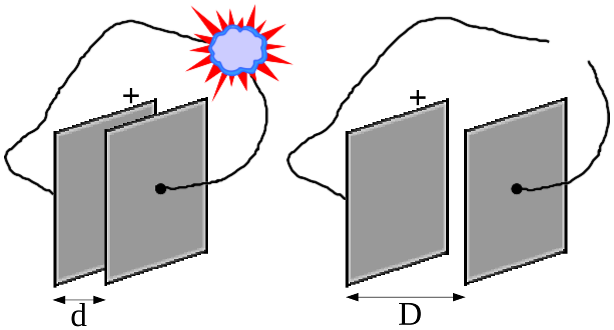
So, in charging a capacitor from $0 \rightarrow Q$ the *total* increase in potential energy is:

$$U = \int dU = \int_0^Q \left(\frac{q}{C}\right)dq = \frac{1}{2C}\left[q^2\right]_0^Q = \frac{1}{2}\frac{Q^2}{C}$$

$$\left(= \frac{1}{2}QV = \frac{1}{2}CV^2\right).$$

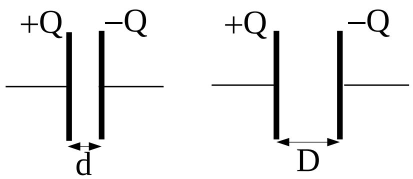


Note, U is the area under the V – Q plot. This potential energy can be recovered when the capacitor is *discharged*, i.e., when the stored charge is released. (Note also, this is the same expression we obtained earlier for a charged conducting sphere.)



Question 24.3: A parallel plate capacitor, with a plate separation of d, is charged by a battery. After the battery is disconnected, the capacitor is discharged through two wires producing a spark. The capacitor is re-charged exactly as before. After the battery is disconnected, the plates are pulled apart slightly, to a new distance D (where D > d). When discharged again, is the energy of the spark_____________it was before the plates were pulled apart?

A: greater than
B: the same as
C: less than



[1] *Algebraically*: Put $C_1 = \varepsilon_0 \dfrac{A}{d}$ and $C_2 = \varepsilon_0 \dfrac{A}{D}$

Since D > d, then $C_2 < C_1$. The charge must be the same in each case (*where can it go or come from?*)

$$\therefore U_1 = \frac{1}{2}\frac{Q^2}{C_1} \text{ and } U_2 = \frac{1}{2}\frac{Q^2}{C_2},$$

i.e., $U_2 > U_1$.

Therefore, the stored energy *increases* when the plates are pulled apart so the spark has more energy.
Answer A.

[2] *Conceptually*: You do (positive) work to separate the plates (because there is an attractive force between them). The work goes into the capacitor system so the stored energy *increases*.
Answer A.

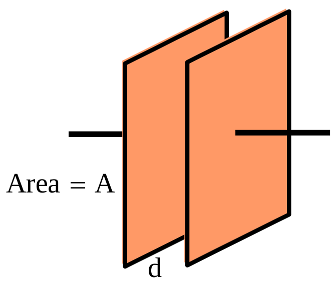
Energy density of an electric field ...

Assume we have a parallel plate capacitor, then the stored energy is:

$$U = \frac{1}{2} C V^2.$$

But $C = \varepsilon_0 \frac{A}{d}$ and $V = E.d$,

$$\therefore U = \frac{1}{2} \varepsilon_0 \frac{A}{d} (E.d)^2 = \frac{1}{2} \varepsilon_0 (A.d) E^2.$$



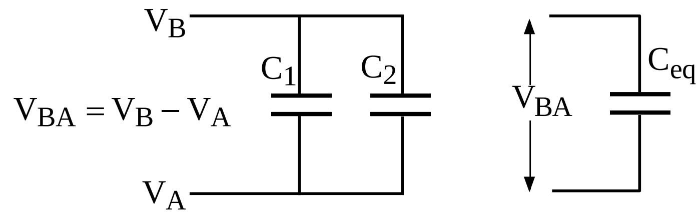
But $A.d \Rightarrow$ volume of the electric field between the plates.

So the *energy density* is:

$$u_e = U/\text{volume} = \frac{1}{2} \varepsilon_0 E^2.$$

This result is true for *all* electric fields.

Combining capacitors (parallel):



Equivalent capacitor

The potential difference is the same on each capacitor. The charges on the two capacitors are:

$$Q_1 = C_1 V_{BA} \text{ and } Q_2 = C_2 V_{BA}$$

and the total charge stored is:

$$Q = Q_1 + Q_2 = (C_1 + C_2) V_{BA} = C_{eq} V_{BA}$$

where $C_{eq} = C_1 + C_2$.

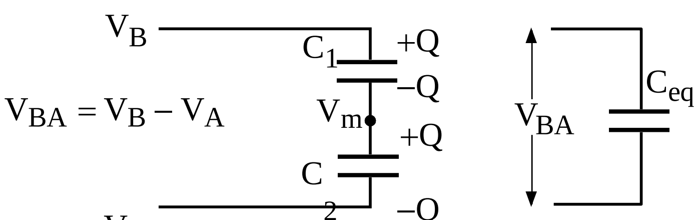
So, this combination is equivalent to a single capacitor with capacitance

$$C_{eq} = C_1 + C_2.$$

When more than two capacitors are connected in parallel:

$$C_{eq} = \sum_i C_i.$$

Combining capacitors (series):



Equivalent capacitor

The charge on the two capacitors is the same: $\pm Q$.

If $V_B > V_A$, the individual potential differences are:

$$V_1 = (V_B - V_m) = \frac{Q}{C_1} \quad \text{and} \quad V_2 = (V_m - V_A) = \frac{Q}{C_2}.$$

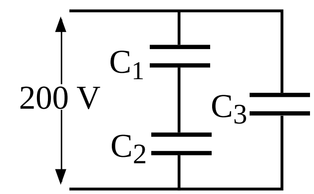
Therefore, the total potential difference is:

$$\therefore V_{BA} = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{Q}{C_{eq}},$$

providing $\dfrac{1}{C_{eq}} = \dfrac{1}{C_1} + \dfrac{1}{C_2}$.

With more than two capacitors:

$$\frac{1}{C_{eq}} = \sum_i \frac{1}{C_i}.$$



$C_1 = 4.0\ \mu F$
$C_2 = 15.0\ \mu F$
$C_3 = 12.0\ \mu F$

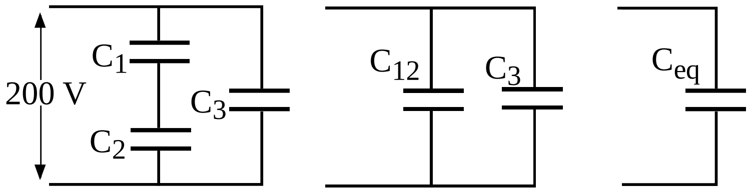
Question 24.4: In the circuit shown above, the capacitors were completely discharged before being connected to the voltage source. Find

(a) the equivalent capacitance of the combination,

(b) the charge on the positive plate of each capacitor,

(c) the potential difference (voltage) across each capacitor, and

(d) the energy stored in each capacitor.



(a) Note that C_1 and C_2 are in series:

$$\therefore \frac{1}{C_{12}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{4\ \mu F} + \frac{1}{15\ \mu F} = 0.3167 \times 10^6.$$

$$\therefore C_{12} = 3.16\ \mu F.$$

But C_{12} and C_3 are in parallel:

$$\therefore C_{eq} = C_{12} + C_3 = 3.16\ \mu F + 12\ \mu F = 15.16\ \mu F.$$

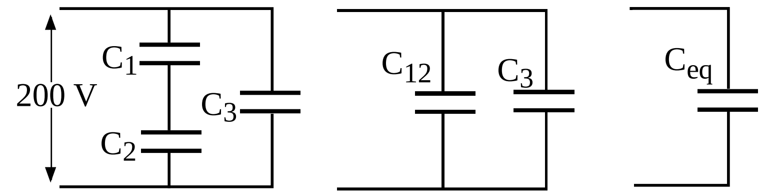
(b) We have: $Q_1 = Q_2$ $(= Q)$

and $V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$

$$\therefore 200 = Q \times 0.3167 \times 10^6,$$

i.e., $Q = \frac{200}{0.3167 \times 10^6} = 0.632 \times 10^{-3} C$ $(= Q_1 = Q_2)$

Also $Q_3 = C_3 V = 12 \times 10^{-6} \times 200 = 2.4 \times 10^{-3} C.$



(c) By definition $V_i = \frac{Q_i}{C_i}$.

$$\left.\begin{aligned} \therefore V_1 &= \frac{Q_1}{C_1} = \frac{0.632 \times 10^{-3}}{4 \times 10^{-6}} = 158\ V, \\ V_2 &= \frac{Q_2}{C_2} = \frac{0.632 \times 10^{-3}}{15 \times 10^{-6}} = 42\ V, \end{aligned}\right\} 200\ V$$

and $V_3 = 200\ V$.

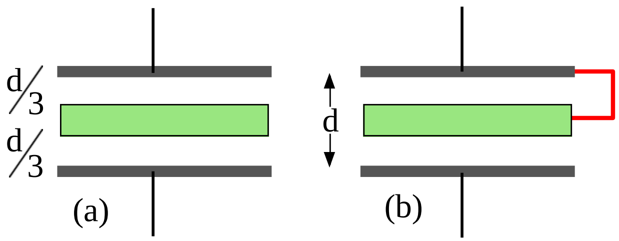
(d) By definition $U_i = \frac{1}{2} Q_i V_i$,

$$\therefore U_1 = 0.05\ J : U_2 = 0.013\ J : U_3 = 0.24\ J.$$

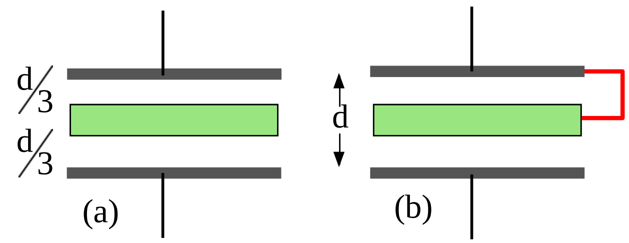
Check total stored energy using the equivalent ...

$$U = \frac{1}{2} C_{eq} V^2 = \frac{1}{2} \times 15.16 \times 10^{-6} \times 200^2 = 0.303\ J,$$

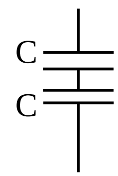
i.e., the same.



Question 24.5: Two capacitors each have a plate separation d. A slab of metal is placed between the plates as shown. In case (a) the slab is not connected to either plate; in case (b) it is connected to the upper plate. Which arrangement produces the higher capacitance, or do they have the same capacitance?



Case (a) is equivalent to two capacitors in series each with capacitance C and spacing $d/3$:



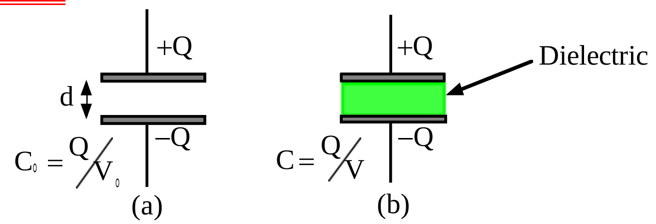
$$\frac{1}{C_{eq}} = \frac{1}{C} + \frac{1}{C} = \frac{2}{C}$$

$$\therefore C_{eq} = \frac{C}{2}.$$

Case (b) is a single capacitor: $C_{eq} = C$.

Therefore, (b) has the higher capacitance.

Dielectrics:



(a) The electric field in an isolated charged parallel plate capacitor (in vacuum) is: $E_0 = \dfrac{\sigma}{\varepsilon_0}$.

$$\therefore V_0 = E_0 d = \frac{\sigma}{\varepsilon_0} d.$$

(b) When a dielectric is inserted, $\varepsilon_0 \Rightarrow \kappa\varepsilon_0$, where is the *dielectric constant*, then $E = \dfrac{\sigma}{\kappa\varepsilon_0}$.

$$\therefore V = Ed = \frac{\sigma}{\kappa\varepsilon_0} d = \frac{V_0}{\kappa},$$

i.e., the potential difference is *reduced*, but the charge remains the same.

$$\therefore C = \frac{Q}{V} = \frac{\kappa Q}{V_0} = \kappa C_0 \quad \left(= \kappa\varepsilon_0 \frac{A}{d}\right).$$

Thus, the capacitance *increases* by a factor of .

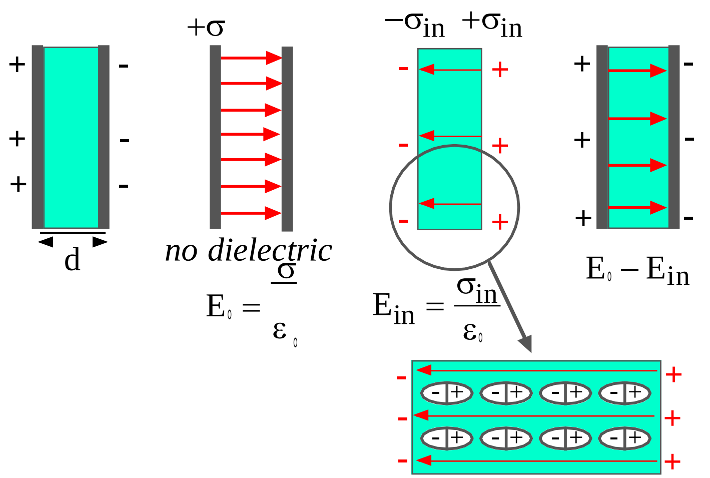
Dielectrics (continued):

Three advantages:

- maintains plate separation when small,
- increased capacitance for a given size.
- dielectric increases the max. electric field possible, and hence potential difference (voltage), between plates before breakdown (*dielectric strength*).

Material		Dielectric Strength (V/m)
Air	1.00059	3×10^6
Paper	3.7	16×10^6
Neoprene	6.9	12×10^6
Polystyrene	2.55	24×10^6

How does a dielectric work? ...



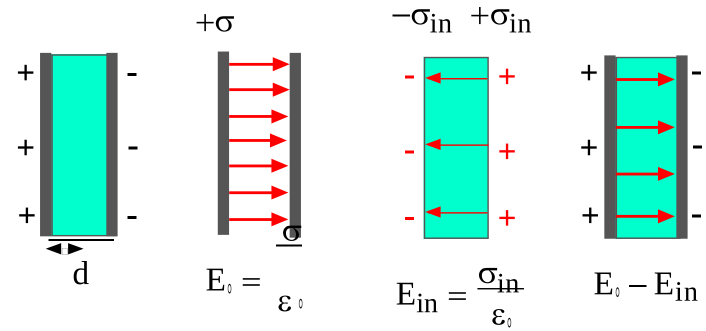
dielectric becomes polarized

Because the dielectric is polarized, the electric field in the presence of a dielectric is *reduced* to:

$$E = E_0 - E_{in} = \frac{\sigma}{\varepsilon_0} - \frac{\sigma_{in}}{\varepsilon_0} = \frac{\sigma - \sigma_{in}}{\varepsilon_0} = \frac{E_0}{\kappa}.$$

Therefore, the potential difference V (= Ed) is reduced by a factor of   also. Since $C \propto 1/V$

$$C > C_0.$$



Hence, the induced electric field is:

$$E_{in} = E_0 - \frac{E_0}{\kappa} = \left(\frac{\kappa - 1}{\kappa}\right) E_0.$$

Substituting for E_{in} and E_0, we obtain

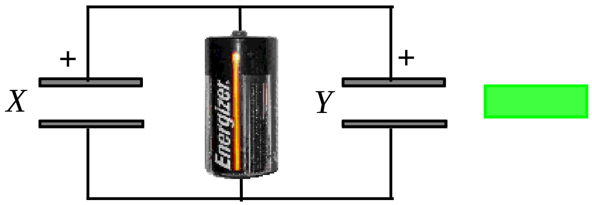
$$\sigma_{in} = \left(\frac{\kappa - 1}{\kappa}\right)\sigma.$$

Hence $\sigma_{in} \le \sigma$. Note: $\sigma_{in} = 0$ if $\kappa = 1$ (*free space*, i.e., no dielectric).

For a conducting slab   and $\sigma_{in} = \sigma$.

$$\therefore E = E_0 - E_{in} = 0,$$

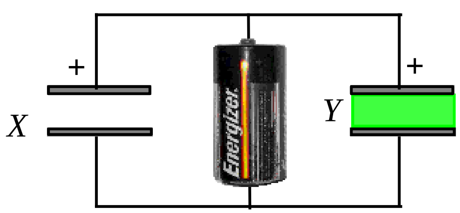
i.e., there is no electric field inside a conductor.



Question 24.7: Two, identical capacitors, *X* and *Y*, are connected across a battery as shown. A slab of dielectric is then inserted between the plates of *Y*.

(a) Which capacitor has the greater charge or do the charges remain the same?

(b) What difference (if any) would it make if the battery was disconnected before the dielectric was inserted?



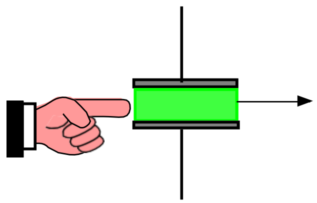
(a) When the dielectric is inserted, the capacitance of *Y* <u>increases</u> from C to κC, where is the dielectric constant. But the capacitance of *X* is <u>unchanged</u>. The potential difference across both capacitors remains the same (= V) and since the charge on a capacitor is given by Q = CV, if C increases to κC, then Q increases to κQ.

Where does the extra charge come from? ...

... from the battery!

(b) If the battery was disconnected before the dielectric was inserted, the charge on each capacitor is unchanged. But, since the capacitance of *Y* increases to κC, the potential difference across *Y* changes from

$$Q/C \text{ to } Q/\kappa C,$$ i.e., it gets smaller.



<u>Question 24.8</u>: A dielectric is placed between the plates of a capacitor. The capacitor is then charged by a battery. After the battery is disconnected, the dielectric is removed.

(a) Does the energy stored by the capacitor increase, decrease or remain the same after the dielectric is removed?

(b) If the battery remains connected when the dielectric is removed, does the energy increase, decrease or remain the same?

Two ways to answer part (a) ...

[1] *Algebraically*: Let $C \to C_0$ and $V \to V_0$

With the dielectric: $U = \frac{1}{2}CV^2$.

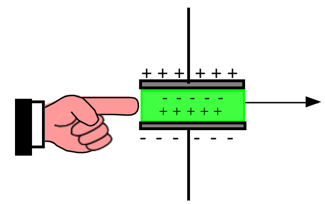
Without the dielectric: $U_0 = \frac{1}{2}C_0 V_0^2$.

But $C = \kappa C_0$, i.e., $C_0 = \frac{C}{\kappa}$, and $V = \frac{V_0}{\kappa}$, i.e., $V_0 = \kappa V$.

$$\therefore U_0 = \frac{1}{2}\left(\frac{C}{\kappa}\right)(\kappa V)^2 = \kappa\left(\frac{1}{2}CV^2\right) = \kappa U.$$

Therefore, the energy *increases.*

[2] *Conceptually*: You must do work to remove the dielectric. Therefore, the stored (potential) energy *increases*!



(b) If the battery remains connected, the potential difference remains constant (= V). But the capacitancechanges from $C \rightarrow C_0$, with $C = \kappa C_0$, i.e., $C > C_0$.

The energy changes from

$$U = \frac{1}{2} C V^2 \text{ to } U_\circ = \frac{1}{2} C_\circ V^2.$$

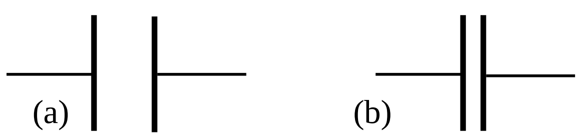
Since $C > C_0$, then $U > U_0$, i.e., the energy decreases.

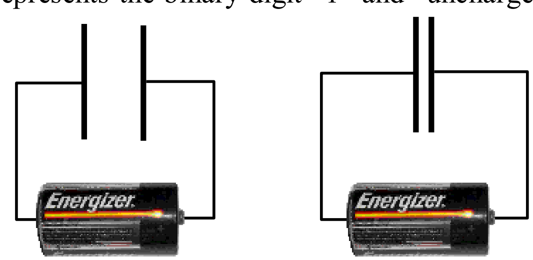
<u>Question 24.10</u>: A parallel plate capacitor is charged bya generator. The generator is then disconnected (a). If the spacing between the plates is decreased (b), what happens to:

(i) the charge on the plates,

(ii) the potential across the plates, and

(iii) the energy stored by the capacitor. of banks of capacitors. A "charged" capacitor represents the binary digit "1" and "uncharged" capacitor represents the binary digit "0".





A parallel plate capacitor is charged by a generator. The generator is then disconnected (a). If the spacing betweenthe plate is decreased (b), what happens to:

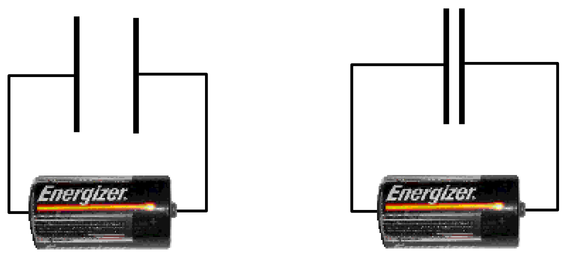
(i) the charge on the plates *<u>remains the same</u>, wherecould it go or where could charge come from?*

(ii) the potential across the plates *<u>decreases</u>, because theelectric field remains constant - it's independent of d (chapter 22) - but as spacing decreases, the potential (V= E.d) decreases.*

(iii) the energy stored by the capacitor *<u>decreases</u>, because the system does work to reduce spacing. Conversely, you would have do work in order increasethe spacing.*

<u>Question 24.11</u>: A parallel plate capacitor is charged bya battery (a). When fully charged, and while the batteryis still connected, the spacing between the plate is decreased (b), what happens to:

(i) the potential across the plates,
(ii) the charge on the plates, and
(iii) the energy stored by the capacitor.



(a) (b)

A parallel plate capacitor is charged by a generator (a). When fully charged, and while the generator is still connected, the spacing between the plate is decreased (b).

(i) the potential across the plates *<u>remains the same</u> since the source of potential difference is still connected!*

(ii) the charge on the plates *<u>increases</u>, because the capacitance increases and so, if V is unchanged, Q increases.*

(iii) the energy stored by the capacitor *<u>increases</u>, because Q and C increase* ($U = \frac{1}{2}QV = \frac{1}{2}CV^2$).