Newton Raphson method:

Given an approximate value of a root of an equation, a better and closer approximation to the root can be found by using an iterative process called Newton's method (or) Newton-Raphson method.

Let α_0 be an approximate value of a root of the equation $f(x)=0$.

Let α be the exact root nearer to α_0. Then $\alpha=\alpha_0+h$ where h is very small positive (or) negative.

$\therefore f(\alpha)=f(\alpha_0+h)=0$ since α is the exact root of $f(x)=0$.

By Taylor expansion,

$$f(\alpha)=f(\alpha_0+h)=f(\alpha_0)+hf'(\alpha_0)+\frac{h^2}{2!}f''(\alpha_0)+\cdots$$

(ie) if h is small, neglecting $h^2, h^3 \ldots$ etc., we get $f(\alpha_0)+hf'(\alpha_0)=0$.

$$\therefore h=\frac{-f(\alpha_0)}{f'(\alpha_0)} \quad \text{if } f'(\alpha_0)\neq 0.$$

$$\therefore \alpha=\alpha_0+h=\alpha_0-\frac{f(\alpha_0)}{f'(\alpha_0)} \text{ approximately}$$

Let this value be α_1.

$$\alpha_1=\alpha_0-\frac{f(\alpha_0)}{f'(\alpha_0)}$$

α_1 is a better approximate root than α_0.

Starting with this α_1 we get

$$\alpha_2 = \alpha_1 - \frac{f(\alpha_1)}{f'(\alpha_1)}$$ which is still better

continuing like this, we iterate this process until $|\alpha_{r+1} - \alpha_r|$ is less than the quantity desired.

$$\therefore \alpha_{r+1} = \alpha_r - \frac{f(\alpha_r)}{f'(\alpha_r)}, \quad r = 0, 1, 2, \ldots$$

This is the iterative formula ~~fo~~ of Newton-Raphson method.

1) Find the positive root of $f(x) = 2x^3 - 3x - 6 = 0$ by Newton-Raphson method correct to five decimal places.

Soln: Let $f(x) = 2x^3 - 3x - 6$, $f'(x) = 6x^2 - 3$

$f(1) = 2 - 3 - 6 = -7 = -ve$

& $f(2) = 16 - 6 - 6 = 4 = +ve$

$\therefore$ a root lies between 1 & 2.

Take $\alpha_0 = 2$

$$\therefore \alpha_{i+1} = \alpha_i - \frac{f(\alpha_i)}{f'(\alpha_i)}$$

$$= \alpha_i - \left[\frac{2\alpha_i^3 - 3\alpha_i - 6}{6\alpha_i^2 - 3}\right]$$

$$= \frac{6\alpha_i^3 - \cancel{3\alpha_i} - 2\alpha_i^3 + \cancel{3\alpha_i} + 6}{6\alpha_i^2 - 3}$$

$$\therefore \alpha_{i+1} = \frac{4\alpha_i^3 + 6}{6\alpha_i^2 - 3}$$

$$\alpha_1 = \frac{4(2)^3 + 6}{6(2)^2 - 3} = \frac{38}{21}$$

$$\alpha_1 = 1.809524$$

$$\alpha_2 = \frac{4(1.809524)^3 + 6}{6(1.809524)^2 - 3} = 1.784200$$

$$\alpha_3 = \frac{4(1.784200)^3 + 6}{6(1.784200)^2 - 3} = 1.783769$$

$$\alpha_4 = \frac{4(1.783769)^3 + 6}{6(1.783769)^2 - 3} = 1.783769$$

The better approximate root is $\underline{1.783769}$ //

2) Find the real positive root of $3x - \cos x - 1 = 0$ by Newton's method correct to 6 decimal places.

Soln: Let $f(x) = 3x - \cos x - 1 \Rightarrow f'(x) = 3 + \sin x$.

$f(0) = 0 - 1 - 1 = -2 = -ve$

$f(0.5) = -1.727582562 = -ve$.

$f(1) = 1.459697694 = +ve$.

Take $\alpha_0 = 1$

$$\therefore \alpha_1 = \alpha_0 - \frac{f(\alpha_0)}{f'(\alpha_0)}$$

$$= \alpha_0 - \frac{(3\alpha_0 - \cos\alpha_0 - 1)}{3 + \sin}$$

$$\alpha_{i+1} = \alpha_i - \frac{f(\alpha_i)}{f'(\alpha_i)}$$

$$= \alpha_i - \frac{(3\alpha_i - \cos\alpha_i - 1)}{3 + \sin\alpha_i}$$

$$= \frac{3\alpha_i + \alpha_i \sin\alpha_i + \cos\alpha_i + 1 - 3\alpha_i}{3 + \sin\alpha_i}$$

$$\therefore \alpha_{i+1} = \frac{\alpha_i \sin\alpha_i + \cos\alpha_i + 1}{3 + \sin\alpha_i}$$

$$\therefore \alpha_1 = \frac{\alpha_0 \sin\alpha_0 + \cos\alpha_0 + 1}{3 + \sin\alpha_0}$$

$$\alpha_1 = \frac{2.381773291}{3.841470985} = 0.620015952$$

Ily $\alpha_2 = 0.60712066$

$\alpha_3 = 0.607101648$

$\alpha_4 = 0.607101648$

$\therefore$ The root is 0.607102 correct to six decimals.

i) Find an iterative formula to find $\sqrt{N}$ (where N is a positive number) and hence find $\sqrt{5}$.

ii) Find an iterative formula to find the reciprocal of a given number N and hence find the value of $\frac{1}{19}$.

) Find the positive root of $x = \cos x$ using Newton's method.

Problem

1. Find the Positive root of $x^4 - x = 10$

Ans

$f(x) = x^4 - x - 10 \Rightarrow f'(x) = 4x^3 - 1$

$f(0) = -10$

$f(1) = 1 - 1 - 10 = -10 = -ve$

$f(2) = 16 - 2 - 10 = 24 = +ve$

The root lies b/w is 1 and 2

Take $x_0 = 2$

By NR Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(x) = 4x^3 - 1$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{((2)^4 - 2 - 10)}{4(2)^3 - 1}$$

$$= 2 - \frac{4}{31}$$

$$x_1 = 1.871$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.871 - \frac{((1.871)^4 - 1.871 - 10)}{4(1.871)^3 - 1}$$

$= 1.871 - \frac{0.383}{25.179}$

$= 1.8557$

$x_2 = 1.856$ ✓

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \Rightarrow 1.856 - \frac{(1.856)^4 - 1.856 - 10}{4(1.856)^3 - 1}$

$= 1.856 - \frac{0.010}{24.574}$

$x_3 = 1.856$ ✓

Hence the root is 1.856.

2. Using NR Method Solve $x \log_{10} x = 12.34$. Start with $x_0 = 10$.

Soln

$$f(x) = x \log_{10} x - 12.34$$

$$f'(x) = x \frac{1}{x} \log_{10} e + \log_{10} x$$

$$= \log_{10} e + \log_{10} x$$

Given $x_0 = 10$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 10 - \frac{10 \log_{10} 10 - 12.34}{\log_{10} e + \log_{10} 10}$$

$$= 10 - \left[\frac{-2.34}{1.4343}\right]$$

$$= 10 + \frac{2.34}{1.4343}$$

$$= 11.6315$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 11.6315 - \left[\frac{1.6315 \log_{10} 11.6315 - 12.34}{\log_{10} e + \log_{10} 11.6315}\right]$$

$$= 11.6315 - \frac{0.0549}{1.5}$$

$$= 11.5949$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 11.5949 - \frac{11.5949\log_{10}11.5949 - 12.34}{\log_{10}e + \log_{10}11.5949}$$

$$= 11.5949 - \frac{0.00006}{1.4986}$$

$$= 11.5949$$

Hence the root is 11.5949

07/02/23 Thursday.

Gauss Jordan Method:

1. State the principle used in Gauss Jordan method

Coefficient matrix is transformed into diagonal Matrix.

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1. Solve the System of Equation by Gauss Jordan Method.

$$5x+4y=15$$
$$3x+7y=12$$

Ans

The Given system is equalent to

$$\cancel{\left[\begin{array}{cc|c} 5 & 4 & 15 \\ 3 & 7 & 12 \end{array}\right]}$$

$$\begin{pmatrix} 5 & 4 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \end{pmatrix}$$

Auguemented Matrix

$$(A,B) \sim \left[\begin{array}{cc|c} 5 & 4 & 15 \\ 3 & 7 & 12 \end{array}\right]$$

$$\sim \left(\begin{array}{cc|c} 5 & 4 & 15 \\ 0 & 23 & 15 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to 5R_2 - 3R_1 \end{array}$$

$5(3)-3(5)=0$

$5(7)-3(4)=35-12=23$

$5(12)-3(15)=60-45=15$

$$\sim \left(\begin{array}{cc|c} 115 & 0 & 285 \\ 0 & 23 & 15 \end{array}\right) \begin{array}{l} R_1 \to 23R_1 - 4R_2 \\ R_2 \end{array}$$

$R_1 \to 23R_1 - 4R_2$

$\to 23(5)-4(0) \Rightarrow 115$

$115x = 285$

$x = \frac{285}{115}$

$\boxed{x = 2.47}$

$23y = 15$

$y = 15/23$

$y = 0.65.$

2. Solve.

$$x + 3y + 3z = 16$$
$$x + 4y + 3z = 18$$
$$x + 3y + 4z = 19$$

by Gaum Jordon Method,

Solve

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16 \\ 18 \\ 19 \end{pmatrix}$$

$$(A/B) \sim \left(\begin{array}{ccc|c} 1 & 3 & 3 & 16 \\ 1 & 4 & 3 & 18 \\ 1 & 3 & 4 & 19 \end{array}\right)$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 3 & 16 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$R_1 \to R_1 - 3R_2$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$R_1 \to R_1 - 3R_3$

09/02/23 Thursday

Iterative methods

a) Gauss Jacobi method

b) Gauss seidel method.

1. State a Sufficient Condition Gauss Jacobi to Converge. (or)
Write a Sufficient Condition Gauss Seidal Meth. to Converge

Ans

The co-efficient of matrix should be diagonally dominant.

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2. Solve the following system by Gauss Jacobi & Seidal method.

$$27x + 6y - z = 85$$
$$x + y + 54z = 110$$
$$6x + 15y + 2z = 72$$

Soln

As the co-efficient matrix is not diagonally dominant. rewrite the equation

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110.$$

① $\Rightarrow x = \frac{1}{27}(85 - 6y + z)$

② $\Rightarrow y \Rightarrow \frac{1}{15}(72 - 6x - 2z)$

③ $\Rightarrow z \Rightarrow \frac{1}{54}(110 - x - y)$

(i) Gauss Jacobi Method :

Let the initial Solution the

$x=0,\ y=0,\ z=0.$

$x = \frac{85}{27}$, $y = \frac{72}{15}$, $z = \frac{110}{54}$

Iteration	x	y	z
0	0	0	0
1	3,148	4.8	2,037
2	2.156	3,269	1.889
3	2,491	3,685	1.936
4	2.400	3.545	1.922
5.	2,431	3,583	1.926.
6.	2,423	3,570	1.925
7.	2,426	3.574	1,926
8.	2,425	3.572	1,925
9.	2,425	3,573	1.925
10.	2.425	3.573	1.925

Gauss-Siedel method:

This method is only a refinement of Gauss-Jacobi method. As before

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \quad \text{---} (1)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \quad \text{---} (2)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \quad \text{---} (3)$$

We start with the initial values $y^{(0)}, z^{(0)}$ for y & z & get $x^{(1)}$ from the 1st eqn.

(ie) $x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$

While using the second equation, we use $z^{(0)}$ for z & $x^{(1)}$ for x instead of $x^{(0)}$ as in the Jacobi's method, we get

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(0)})$$

Now, having known $x^{(1)}$ & $y^{(1)}$, use $x^{(1)}$ for x & $y^{(1)}$ for y in the third equation, we get

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

To find the values of the unknowns, we use the latest available values on the Right hand side.

If $x^{(r)}, y^{(r)}, z^{(r)}$ are the r^{th} iterates, then the iteration scheme will be

$$x^{r+1} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{r+1} = \frac{1}{b_1}(d_2 - a_2 x^{(r+1)} - c_2 z^{(r)})$$

$$z^{r+1} = \frac{1}{c_3}(d_3 - a_3 x^{(r+1)} - b_3 z^{(r+1)})$$

This process of iteration is continued until the convergence is confirmed.

Ex (1) Solve the following by Gauss-Jacobi and Gauss-Seidel method:

$$10x - 5y - 2z = -3$$

$$2x - 10y + 3z = -3 \Rightarrow -4x + 10y + 3z = 3$$

$$x + 6y + 10z = -3$$

Solu. Here the diagonal elements are dominant, Hence, the iteration process can be applied.

That is, the coefficient matrix $\begin{pmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{pmatrix}$ is diagonally dominant,

since $|10| > |-5| + |-2|$,

$|-10| > |4| + |3|$ 8

$|10| > |1| + |6|$

Gauss Jacobi method:

Solving x, y, z, we have

$$x = \frac{1}{10}(3 + 5y + 2z) \quad \text{---} \quad (1)$$

$$y = \frac{1}{10}(3 + 4x + 3z) \quad \text{---} \quad (2)$$

$$z = \frac{1}{10}(-3 - x - 6y) \quad \text{---} \quad (3)$$

First iteration:

Let the initial values be $(0, 0, 0)$. Using these initial values in (1), (2), (3), we get

$$x^{(1)} = \frac{1}{10}[3 + 5(0) + 2(0)] = 0.3$$

$$y^{(1)} = \frac{1}{10}[3 + 4(0) + 3(0)] = 0.3$$

$$z^{(1)} = \frac{1}{10}[-3 - (0) - 6(0)] = -0.3$$

2nd iteration.

Using these values in (1), (2) & (3) we get

$$x^{(2)} = \frac{1}{10}[3 + 5(0.3) - 2(-0.3)] = 0.39$$

$$y^{(2)} = \frac{1}{10}[3 + 4(0.3) + 3(-0.3)] = 0.33$$

$$z^{(2)} = \frac{1}{10}[-3 - (0.3) - 6(0.3)] = -0.51$$

Third iteration:

Using the values of $x^{(2)}, y^{(2)}, z^{(2)}$ in (1), (2), (3)

$$x^{(3)} = \frac{1}{10}[3+5(0.33)+2(-0.51)] = 0.363$$

$$y^{(3)} = \frac{1}{10}[3+4(0.39)+3(-0.51)] = 0.303$$

$$z^{(3)} = \frac{1}{10}[-3-(0.39)-6(0.33)] = -0.537$$

Fourth iteration:

$$x^{(4)} = \frac{1}{10}[3+5(0.303)+(-0.537)] = 0.3441$$

$$y^{(4)} = \frac{1}{10}[3+4(0.363)+3(-0.537)] = 0.2841$$

$$z^{(4)} = \frac{1}{10}[-3-0.363-6(0.303)] = -0.5181$$

Fifth iteration:

$$x^{(5)} = \frac{1}{10}[3+5(0.2841)+2(-0.5181)] = 0.33843$$

$$y^{(5)} = \frac{1}{10}[3+4(0.3441)+3(-0.5181)] = 0.2822$$

$$z^{(5)} = \frac{1}{10}[-3-(0.3441)-6(0.2841)] = -0.50487$$

Sixth iteration:

$$x^{(6)} = \frac{1}{10}[3+5(0.2822)+2(-0.50487)] = 0.3840126$$

$$y^{(6)} = \frac{1}{10}[3+4(0.33843)+3(-0.50487)] = 0.283911$$

$$z^{(6)} = \frac{1}{10}[-3-(0.33843)-6(0.2822)] = -0.503163$$

Seventh iteration:

$$x^{(7)} = \frac{1}{10}[3+5(0.283911)+2(-0.503163)] = 0.3413229$$

$$y^{(7)} = \frac{1}{10}[3+4(0.340126)+3(-0.53163)] = 0.2851015$$

$$z^{(7)} = \frac{1}{10}[-3-(0.340126)-6(0.283911)] = -0.5043592$$

Eighth iteration:

$$x^{(8)} = \frac{1}{10}[3+5(0.2851015)+2(-0.5043592)] = 0.34167809$$

$$y^{(8)} = \frac{1}{10}[3+4(0.3413229)+3(-0.5043592)] = 0.2852214$$

$$z^{(8)} = \frac{1}{10}[-3-(0.3413229)-6(0.2851015)] = -0.50519319.$$

Nineth iteration:

$$x^{(9)} = \frac{1}{10}[3+5(0.2852214) + 2(-0.50519319)] = 0.341572062$$

$$y^{(9)} = \frac{1}{10}[3+4(0.3416787) + 3(-0.50519319)] = 0.285113607$$

$$z^{(9)} = \frac{1}{10}[-3-(0.3416789) - 6(0.2852214)] = -0.50530073$$

Hence correct to 3 decimal places, the values are

$x = 0.342$, $y = 0.285$, $z = -0.505$

Gauss-Seidel method:

initial values $y=0$, $z=0$

First iteration:

$$x^{(1)} = \frac{1}{10}[3+5(0) + 2(0)] = 0.3$$

$$y^{(1)} = \frac{1}{10}[3+4(0.3) + 3(0)] = 0.42$$

$$z^{(1)} = \frac{1}{10}[-3-(0.3) - 6(0.42)] = -0.582$$

Second iteration:

$$x^{(2)} = \frac{1}{10}[3+5(0.42) + 2(-0.582)] = 0.3936,$$

$$y^{(2)} = \frac{1}{10}[3+4(0.3936) + 3(-0.582)] = 0.28284$$

$$z^{(2)} = \frac{1}{10}[-3-(0.3936) - 6(0.28284)] = -0.509064.$$

Third iteration:

$$x^{(3)} = \frac{1}{10}[3+5(0.28284) + 2(-0.509064)] = 0.3396072$$

$$y^{(3)} = \frac{1}{10}[3+4(0.3396072) + 3(-0.509064)] = 0.28312368$$

$$z^{(3)} = \frac{1}{10}[-3-(0.3396072) - 6(0.28312368)] = -0.50383492$$

Fourth iteration:

$$x^{(4)} = \frac{1}{10}[3+5(0.28312368) + 2(-0.50383492)] = 0.34079485$$

$$y^{(4)} = \frac{1}{10}[3+4(0.34079489) + 3(-0.50383492)] = 0.28516746.$$

$$z^{(4)} = \frac{1}{10}[-3-(0.34079485) - 6(0.28516746)] = -0.50517996.$$

Fifth iteration:

$$x^{(5)} = \frac{1}{10}[3+5(0.28516746)+2(-0.50517996)] = 0.3415547$$

$$y^{(5)} = \frac{1}{10}[3+4(0.34155477)+3(-0.50517996)] = 0.28506792$$

$$z^{(5)} = \frac{1}{10}[-3-(0.34155477)-6(0.28506792)] = -0.505196229.$$

Sixth iteration:

$$x^{(6)} = \frac{1}{10}[3+5(0.28506792)+2(-0.505196229)] = 0.341494714$$

$$y^{(6)} = \frac{1}{10}[3+4(0.341494714)+3(-0.505196229)] = 0.285039017$$

$$z^{(6)} = \frac{1}{10}[-3-(0.341494714)-6(0.285039017)] = -0.5051728.$$

Seventh iteration:

$$x^{(7)} = \frac{1}{10}[3+5(0.285039017)+2(-0.5051728)] = 0.3414849$$

$$y^{(7)} = \frac{1}{10}[3+4(0.3414849)+3(-0.5051728)] = 0.28504212$$

$$z^{(7)} = \frac{1}{10}[-3-(0.3414849)-6(0.28504212)] = -0.5051737.$$

The values at each iteration by both methods are tabulated below.

Iteration	Gauss-~~Jacobi~~ method x	y	z	Gauss-Seidel method x	y	z
1	0.3	0.3	-0.3	0.3	0.42	-0.582
2	0.39	0.33	-051	0.3936	0.28284	-0.509064
3	0.363	0.303	-0.537	0.3396072	0.28312364	-0.50383492
4	0.3441	0.2841	-0.5181	0.34079485	0.28516746	-0.50517996
5	0.33843	0.2822	-0.50487	0.3415547	0.28506792	-0.505196229
6	0.340126	0.283911	-0.503163	0.3414947	0.2850390	-0.5051728
7	03413229	0.2851015	-0.5043592	0.3414849	0.28504212	-0.5051737
8	0.34167891	0.2852214	-0.50519319			
9	0.341572062	0.28511360	-0.50530073			

The values correct to 3 decimal places are $x = 0.342$, $y = 0.285$, $z = -0.5$

Inverse of a matrix by Gauss Jordan method:

By Gauss-Jordan method, the inverse matrix is obtained by the following steps.

Step:1 First consider the augmented matrix $(A|I)$

Step:2 Reduce the matrix A in $(A|I)$ to the identity matrix I by applying row transformation.

The row transformations used in step 2 transform I to A^{-1}

Finally write the inverse matrix A^{-1}

(ie) $(A|I) \xrightarrow{\text{Gauss-Jordan}} (I|A^{-1})$.

Problem:

1) Using Gauss-Jordan method, find the inverse of the matrix $\begin{bmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$

Soln:

$$(A,I) = \left[\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3/2 & 1/2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \quad R_1 \to R_1/2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3/2 & -1/2 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & 2 & 7/2 & -1/2 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 2 & 7/2 & -1/2 & 0 & 1 \end{array}\right] R_2(-1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & -1/2 & -5/2 & 2 & 1 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ R_2 \\ R_3 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 5 & -4 & -2 \end{array}\right] R_3(-2)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -9 & 7 & 4 \\ 0 & 0 & 1 & 5 & -4 & -2 \end{array}\right] \begin{array}{l} R_1 + \frac{1}{2}R_3 \\ R_2 - 2R_3 \\ \end{array}$$

$$\sim [I, A^{-1}]$$

Hence $A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ -9 & 7 & 4 \\ 5 & -4 & -2 \end{bmatrix}$

2) Using Gauss-Jordan method, find the inverse of

$$\begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 5 \\ 1 & -1 & 0 \end{bmatrix}$$

Solu:

We write augmented matrix (A, I) as

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 5 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

We will bring 1 in the place of a_{11} by interchanging row 1 & row 3.

$$(A, I) \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 1 \\ 3 & 2 & 5 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 5 & 5 & 0 & 1 & -3 \\ 0 & 2 & 1 & 1 & 0 & -2 \end{array}\right] \begin{array}{l} R_1 \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1/5 & -3/5 \\ 0 & 2 & 1 & 1 & 0 & -2 \end{array}\right] R_2\left(\frac{1}{5}\right).$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & \frac{1}{5} & 2/5 \\ 0 & 1 & 1 & 0 & 1/5 & -3/5 \\ 0 & 0 & -1 & 1 & -2/5 & -4/5 \end{array}\right] \begin{array}{l} R_1+R_2 \\ R_2 \\ R_3-2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1/5 & 2/5 \\ 0 & 1 & 1 & 0 & 1/5 & -3/5 \\ 0 & 0 & 1 & -1 & 2/5 & 4/5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/5 & -2/5 \\ 0 & 1 & 0 & 1 & -1/5 & -7/5 \\ 0 & 0 & 1 & -1 & 2/5 & 4/5 \end{array}\right] \begin{array}{l} R_1-R_3 \\ R_2-R_3 \\ R_3 \end{array}$$

$$\sim [I, A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -1/5 & -2/5 \\ 1 & -1/5 & -7/5 \\ -1 & 2/5 & 4/5 \end{bmatrix}$$

Eigenvalue of a matrix by power method.

Power method:

Power method is used to determine numerically largest eigen value and the corresponding eigen vector of a matrix A.

Note: Working rule:

(1) Assume the initial vector x_0.

(2) Then find $Y_{k+1} = AX_k$, $k = 0, 1, 2 \ldots$

(3) Normalize the vector Y_{k+1} to get a new vector X_{k+1}.

(ie) $X_{k+1} = \frac{1}{m_{k+1}} Y_{k+1}$ where m_{k+1} is the largest component in magnitude of Y_{k+1}.

(4) Repeat steps (2) & (3) till convergence is acheived.

Note:

To find the numerically smallest eigen value of A, obtain the dominant eigen value λ_1 of A and then find $B = A - \lambda_1 I$ and find the dominant eigen value of B. Then the smallest eigen value of A is equal to the dominant eigenvalue of $B + \lambda_1$.

Ex(1) Find the dominant eigen value of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by power method and hence find the other eigen value also. Verify your results by any other matrix theory.

Soln:

Let an initial arbitrary vector be $x_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$A x_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} = 4 X_2$$

$$A X_2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 5.5 \end{pmatrix}$$

$$= 5.5 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} = 7.5 X_3$$

$$A X_3 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 7/3 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 7/15 \\ 1 \end{pmatrix} = 5 X_4$$

$$A X_4 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 7/15 \\ 1 \end{pmatrix} = \begin{pmatrix} 37/15 \\ \frac{81}{15} \end{pmatrix} = \frac{81}{15} \begin{pmatrix} \frac{37}{81} \\ 1 \end{pmatrix} = \frac{81}{15}$$

$$A X_5 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0.4568 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.4568 \\ 5.3704 \end{pmatrix}$$

$$= 5.3704 \begin{pmatrix} 0.4575 \\ 1 \end{pmatrix} = 5.3704 X_6$$

$$A x_6 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0.4575 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.4575 \\ 5.3724 \end{pmatrix}$$

$$= 5.3724 \begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = 5.3724\, x_7$$

$$A x_7 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.4574 \\ 5.3723 \end{pmatrix}$$

$$= 5.3723 \begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = 5.3723\, x_8$$

$$A x_8 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.4574 \\ 5.3723 \end{pmatrix} = 5.3723 \begin{pmatrix} 0.4574 \\ 1 \end{pmatrix}$$

Hence $\lambda_1 = 5.3723$ & eigen vector $x_1 = \begin{pmatrix} 0.4574 \\ 1 \end{pmatrix}$

Since $\lambda_1 + \lambda_2 =$ Trace of $A = 1+4 = 5$

$\therefore \lambda_2 = -0.3723$

(ii) Char. equ. is $\lambda^2 - (1+4)\lambda + \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 0$

(ie) $\lambda^2 - 5\lambda = 0$.

$$\therefore \lambda = \frac{5 \pm \sqrt{25+8}}{2} = \frac{5 \pm \sqrt{33}}{2}$$

$$\lambda_1 = 5.3723, \quad \lambda_2 = -0.3723$$

Ex.(2) Find the dominant eigen Value and the Corresponding eigen Vector of $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Find also the least latent root & hence the third eigen Value also.

Sol:

Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an approximate eigen Vector

$$AX_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 . X_2$$

$$AX_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = 7X_3$$

$$AX_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5714 \\ 1.8572 \\ 0 \end{bmatrix}$$

$$= 3.5714 \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix}$$

$$= 3.5714 X_4$$

$$AX_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix}$$

$$= 4.12 \begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = 4.12\, X_5$$

$$AX_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9706 \\ 1.9902 \\ 0 \end{bmatrix}$$

$$= 3.9706 \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = 3.9706\, X_6$$

$$AX_6 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.0072 \\ 2.0024 \\ 0 \end{bmatrix}$$

$$= 4.0072 \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = 4.0072\, X_7$$

$$AX_7 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9982 \\ 1.9994 \\ 0 \end{bmatrix} = 3.9982 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$$= 3.9982\, X_8$$

$$AX_8 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 4\, X_9$$

$\therefore$ Dominant eigen value $= 4$; corresponding eigen vector is $(1, 0.5, 0)$.

To find the least eigen value.

let $B = A - 4I$ since $\lambda_1 = 4$.

$$\therefore B = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

We will find the dominant eigen Value of B.

Let $Y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial vector.

$$BY_1 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -3Y_2$$

$$BY_2 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.6666 \\ 0 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -5Y_3$$

$$\therefore BY_3 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix}$$

$\therefore$ Dominant eigenvalue of B is $= -5$

Adding 4, Smallest eigenValue of $A = -5 + 4 = -1$

Sum of eigen values = Trace of A

$= 1 + 2 + 3 = 6$

$$4 + (-1) + \lambda_3 = 6$$

$$\Rightarrow \lambda_3 = 3$$

$\therefore$ All the three eigen Values are

4, 3, −1