UNIT-II

Interpolation and Approximation

Lagrange's interpolation formula:

Let $y=f(x)$ be a function such that $f(x)$ takes the values $y_0, y_1, y_2 \dots y_n$ corresponding to $x = x_0, x_1, x_2 \dots x_n$.

That is, $y_i = f(x_i)$, $i = 0, 1, 2 \dots n$.

Now, there are $(n+1)$ paired values (x_i, y_i), $i=0,1,2 \dots n$ and hence $f(x)$ can be represented by a polynomial function of degree 'n' in x.

We will select that $f(x)$ as follows.

$$f(x) = a_0 (x-x_1)(x-x_2) \dots (x-x_n)$$
$$+ a_1 (x-x_0)(x-x_2) \dots (x-x_n)$$
$$+ \dots$$
$$+ a_n (x-x_0)(x-x_1) \dots (x-x_{n-1}) \quad \text{---} \quad (1)$$

This is true for all values of x,

Substituting in (1), $x=x_0$, $y=y_0$, we get

$$y_0 = a_0 (x_0-x_1)(x_0-x_2) \dots (x_0-x_n)$$

$$\therefore a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2) \dots (x_0-x_n)}$$

Similarly, setting $x=x_1$, $y=y_1$ we have

$$a_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2) \dots (x_1-x_n)}$$

In the same way, we get

$$a_2 = \frac{y_2}{(x_2-x_0)(x_2-x_1)\cdots(x_2-x_n)}$$

$$\cdots\cdots\cdots\cdots$$

$$a_n = \frac{y_n}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$

Substituting these values of a's in (1), we have

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}\cdot y_0$$

$$+ \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}\cdot y_1$$

$$\cdots\cdots\cdots\cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}\, y_n \quad \text{(2)}$$

Equ. (2) is called Lagrange's interpolation formula for unequal intervals.

(1) Using Lagrange's interpolation formula, find $y(10)$ from the following table.

x :	5	6	9	11
y :	12	13	14	16

Soln: By Lagrange's interpolation formula, we have

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\, y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\, y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\, y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\cdot y_3$$

$$= \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)} \cdot 12 + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)} \cdot 13$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)} \cdot 14 + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)} \cdot 16$$

Putting $x=10$,

$$y(10) = f(10) = \frac{4(1)(-1)}{(-1)(4)(-6)} \cdot 12 + \frac{5(1)(-1)}{(1)(-3)(-5)} \cdot 13 + \frac{(5)(4)(-1)}{4(3)(-2)} \cdot 14 + \frac{(5)(4)(1)}{(6)(5)(2)} \cdot 16$$

$\underline{y(10) = 14.6666}$

2. Find the parabola of the form $y = ax^2+bx+c$ Passing through the pts. $(0,0)$ $(1,1)$ & $(2,20)$

Solu:

By Lagrange's formula, we have

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)} 0 + \frac{(x-0)(x-2)}{(1-0)(1-2)} \cdot 1 + \frac{(x-0)(x-1)}{(2-0)(2-1)} \cdot 20$$

$$= 0 - x(x-2) + 10x(x-1)$$

$$\underline{y = 9x^2 - 8x.}$$

3. The mode of a certain frequency curve $y = f(x)$ is very nearer to $x=9$ and the values of the frequency density $f(x)$ for $x = 8.9, 9, 9.3$ are respectively 0.30, 0.35 & 0.25. Calculate the approximate value of the mode.

Inverse interpolation:

So far, given a set of x & y we were finding the values of y corresponding to some $x = x_k$ (which is not given in the table). Here, we treat y as a function of x. Now the problem is, given some $y = y_r$, we should find the corresponding x. This process of finding x given y is called the inverse interpolation.

In such a case, we will take y as independent variable and x as dependent variable and use Lagrange's interpolation formula.

Taking y as independent variable,

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} \cdot x_0 + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)} x_1$$

$$+ \cdots\cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})} x_n \quad (1)$$

This formula (1) is called formula of inverse interpolation.

(1) From the data given below, find the value of x, when $y = 13.5$

x :	93.0	96.2	100.0	104.2	108.7
y :	11.38	12.80	14.70	17.07	19.91

Inverse interpolation:

So far, given a set of x & y we were finding the values of y corresponding to some $x = x_k$. (which is not given in the table). Here, we treat y as a function of x. Now the problem is, given some $y = y_r$, we should find the corresponding x. This process of finding x given y is called the inverse interpolation.

In such a case, we will take y as independent variable and x as dependent variable and use Lagrange's interpolation formula.

Taking y as independent variable,

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} \cdot x_0 + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)} x_1$$

$$+ \cdots \cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})} x_n \quad (1)$$

This formula (1) is called formula of inverse interpolation.

(1) From the data given below, find the value of x, when $y = 13.5$

x :	93.0	96.2	100.0	104.2	108.7
y :	11.38	12.80	14.70	17.07	19.91

Solu:

By Lagrange's formula for inverse interpolation

$$x = \frac{(y-12.80)(y-14.70)(y-17.07)(y-19.91)}{(11.38-12.80)(11.38-14.70)(11.38-17.07)(11.38-19.91)} \times 93$$

$$+ \frac{(y-11.38)(y-14.70)(y-17.07)(y-19.91)}{(12.80-11.38)(12.80-14.70)(12.80-17.07)(12.80-19.91)} \times 96.2$$

$$+ \frac{(y-11.38)(y-12.80)(y-17.07)(y-19.91)}{(17.07-11.38)(17.07-12.80)(17.07-14.70)(17.07-19.91)} \times 104.2$$

$$+ \frac{(y-11.38)(y-12.80)(y-14.70)(y-17.07)}{(19.91-11.38)(19.91-12.80)(19.91-14.70)(19.91-17.07)} \times 108.7$$

Putting $y = 13.5$ on the RHS, & simplifying

$x = -7.812629 + 68.3721132 + 43.595887 - 7.2733429 + 0.77008419$

$x = 97.6557503$

2) Find the value of θ given $f(\theta) = 0.3887$ where $f(\theta) = \int_0^\theta \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$ Using the table.

θ =	21°	23°	25°
$f(\theta)$ =	0.3706	0.4068	0.4433

Solu:

Now take $f(\theta)$ as independent and θ as dependent

$y = f(\theta)$:	0.3706	0.4068	0.4433
θ:	21	23	25

$$\theta = \frac{(y-0.4068)(y-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)} \times 21$$

$$+ \frac{(y-0.3706)(y-0.4433)}{(0.4068-0.3706)(0.4068-0.4433)} \times 23$$

$$+ \frac{(y-0.3706)(y-0.4068)}{(0.4433-0.3706)(0.4433-0.4068)} \times 25$$

$\theta(y=0.3887)$

$$= \frac{(0.3887-0.4068)(0.3887-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)} \times 21$$

$$+ \frac{(0.3887-0.3706)(0.3887-0.4433)}{(0.4068-0.3706)(0.4068-0.4433)} \times 23$$

$$+ \frac{(0.3887-0.3706)(0.3887-0.4068)}{(0.4433-0.3706)(0.4433-0.4068)} \times 25$$

$$= 7.885832 + 17.202739 - 3.086525$$

$$= 22.0020^{\circ}$$

(3) Find the age corresponding to the annuity value 13.6 given the table,

Age (x) :	30	35	40	45	50
Annuity (y) Value :	15.9	14.9	14.1	13.3	12.5

Divided differences

Let the fun. $y = f(x)$ assume the values $f(x_0), f(x_1) \ldots\ldots f(x_n)$ corresponding to the arguments $x_0, x_1, \ldots\ldots x_n$ respectively where the intervals $x_1 - x_0,\ x_2 - x_1 \ldots\ldots x_n - x_{n-1}$ need not be equal.

The first divided difference of $f(x)$ for the arguments x_0, x_1 is defined as $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$

It is denoted by $f(x_0, x_1)$ (or) $[x_0, x_1]$ (or) $\underset{x_1}{\Delta} f(x_0)$.

In otherwords

$$f(x_0, x_1) = [x_0, x_1] = \underset{x_1}{\Delta} f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \text{---} (1)$$

$$\text{Illy} \quad f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} \quad \text{and so on.}$$

Thus, for defining a first divided difference, we need the functional values, corresponding to two arguments.

The second divided difference of $f(x)$ for three arguments x_0, x_1, x_2 is defined as

$$f(x_0, x_1, x_2) = \underset{x_1, x_2}{\Delta^2} f(x_0) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} \quad \text{---} (2)$$

$$\text{ly} \quad f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} \quad \text{---}$$

The third divided difference of $f(x)$ for the four arguments x_0, x_1, x_2, x_3 is defined as

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0} \quad \text{—— (3)}$$

Equ. (1), (2) & (3) are called divided differences of order one, two & three respectively.

Table

Argument x	Entry $f(x)$	First divided diff. $\Delta f(x)$	2nd divided diff. $\Delta^2 f(x)$	3rd divided diff. $\Delta^3 f(x)$
x_0	$f(x_0)$			
x_1	$f(x_1)$	$f(x_0, x_1)$	$f(x_0, x_1, x_2)$	
x_2	$f(x_2)$	$f(x_1, x_2)$	$f(x_1, x_2, x_3)$	$f(x_0, x_1, x_2, x_3)$
x_3	$f(x_3)$	$f(x_2, x_3)$	$f(x_2, x_3, x_4)$	$f(x_1, x_2, x_3, \ldots$
x_4	$f(x_4)$	$f(x_3, x_4)$		

(1) Form the divided diff. table for the following data

x : -2 0 3 5 7 8

$f(x)$: -792 108 -72 48 -144 -252

Soln:

We form the table below

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-2	-792				
		450	-102		
0	108			18	
		-60			-3
3	-72		24		
		60		-9	
5	48		-34		
		-96			-2
7	-144			7	
		-108	-4		
8	-252				

2. Find the divided diff. of $f(x) = x^3 + x + 2$ for the arguments 1, 3, 6, 11

Soln. Given x: 1 3 6 11

$f(x)$: 4 32 224 1344

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	4			
		14	10	
3	32			1
		64		
6	224		20	
		224		
11	1344			

Properties of divided differences

(1) The value of any divided difference is independent of the order of the arguments. That is, the divided differences are symmetrical in all their arguments.

(2) The operator Δ is linear.

(3) The nth divided differences of a polynomial of degree n are constants.

(3) Show that

$$\underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = \frac{-1}{abcd}$$

Soln:

If $f(x) = \frac{1}{x}$, $f(a) = \frac{1}{a}$

$$f(a,b) = \Delta_b\left(\frac{1}{a}\right) = \frac{\frac{1}{b} - \frac{1}{a}}{b-a}$$

$$= -\frac{1}{ab}$$

$$f(a,b,c) = \frac{f(b,c) - f(a,b)}{c-a} = \frac{-\frac{1}{bc} + \frac{1}{ab}}{c-a}$$

$$= \frac{1}{abc}\left[\frac{c-a}{c-a}\right] = \frac{1}{abc}$$

$$f(a,b,c,d) = \frac{f(b,c,d) - f(a,b,c)}{d-a}$$

$$= \frac{\frac{1}{bcd} - \frac{1}{abc}}{d-a} = -\frac{1}{abcd}$$

$$\therefore \underset{bcd}{\Delta^3}\left[\frac{1}{a}\right] = -\frac{1}{abcd} \;//$$

Newton's interpolation formula for unequal intervals

Let $y = f(x)$ take values $f(x_0), f(x_1) \ldots f(x_n)$ corresponding to the arguments $x_0, x_1 \ldots x_n$

By definition

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\therefore f(x) = f(x_0) + (x - x_0) f(x, x_0) \quad \text{— (1)}$$

Illy $$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

$$\therefore f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1)$$

Using this value of $f(x, x_0)$ in (1), we have

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x, x_0, x_1)$$

Again $$f(x, x_0, x_1, x_2) = \frac{f(x, x_0, x_1) - f(x_0, x_1, x_2)}{x - x_2}$$

$$\therefore f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1, x_2)$$

Using this value in (2), we get

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x, x_0, x_1, x_2) \quad \text{— (3)}$$

Continuing in this manner, we get

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \ldots$$
$$+ (x - x_0)(x - x_1)(x - x_2) \ldots (x - x_{n-1}) f(x_0, x_1, x_2 \ldots x_n)$$
$$+ (x - x_0)(x - x_1)(x - x_2) \ldots (x - x_n) f(x, x_0, x_1, \ldots x_n)$$

(4)

If $f(x)$ is a polynomial of degree n, then $f(x, x_0, x_1, \ldots, x_n) \to 0$ $\{ (n+1)^{th}$ dif $\}$

$\therefore$ (4) becomes.

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) \ldots (x-x_0)(x-x_1)\ldots(x-x_{n-1}) f(x_0, x_1 \ldots x_n) \quad (5)$$

Equ-(5) is called Newton's divided difference interpolation formula for unequal intervals.

Problem:

1. Using Newton's divided difference formula, find the values of $f(2)$, $f(8)$ & $f(15)$ given the following table.

x:	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

Soln: Divided difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
4	48			
		$\frac{100-48}{5-4} = 52$		
5	100		$\frac{97-52}{7-4} = 15$	
		97		1
7	294		21	
		202		1
10	900		27	
		310		1
11	1210		33	
		407		
13	2028			

By Newton's divided diff. interpolation formula,

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + \dots \quad (1)$$

Given $x_0=4$, $x_1=5$, $x_2=7$, $x_3=10$, $x_4=11$ & $x_5=13$

& $f(x_0)=48$, $f(x_0,x_1)=52$, $f(x_0,x_1,x_2)=15$, $f(x_0,x_1,x_2,x_3)=1$

Hence Using these values in (1), we have

$$f(x) = 48 + (x-4)52 + (x-4)(x-5)15 + (x-4)(x-5)(x-7)1$$

$$f(2) = 48 - 104 + 90 - 30 = 4$$

$$f(8) = 48 + (4)52 + (4)(3)15 + 4(3)(1)(1) = 448$$

$$f(15) = 48 + 11 \times 52 + 11 \times 10 \times 15 + 11 \times 10 \times 8 = 3150.$$

2) From the following table find $f(x)$ and hence $f(6)$ using Newton's interpolation formula

x :	1	2	7	8
$f(x)$:	1	5	5	4

Solu: Evidently, intervals are not equal. We form the divided difference table below

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1			
		4		
2	5		$-2/3$	
		0		$\frac{1}{14}$
7	5		$-1/6$	
		-1		
8	4			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x-x_0) f(x_1, x_0) + (x-x_0)(x-x_1) f(x_0,x_1,x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3)$$

$= 1 + (x-1)4 + (x-1)(x-2)\left(-\frac{2}{5}\right) + (x-1)(x-2)(x-7)\left(\frac{1}{41}\right)$

$= \frac{1}{42}(3x^3 - 58x^2 + 31x - 224)$

$f(6) = \frac{1}{42}[3 \times 216 - 36 \times 58 + 1926 - 224]$

$= 6.23809524.$

3 From the following table obtain $f(x)$ as a polynomial in powers of $(x-5)$.

x :	0	2	3	4	5	6
$f(x)$:	4	26	58	112	466	922

using Newton's method.

Solu: We will form the divided difference table below.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0				
	4			
		11		
2			7	
	26			1
3		32		
	58		11	
4		54		1
	112		16	
7		118		1
	466		22	
9		228		$\frac{b-22}{1} = 1$
	922		$\frac{p-228}{-2} = b$	
5		$\frac{a-922}{-4} = p$		$\frac{d-b}{5-7} = 1$
	a		$\frac{q-p}{5-9} = d$	
5		q		
5				$\frac{k-d}{5-9} = 1$
			k	

Since the third differences are constant $(=1)$, we extend the table by introducing $x=5$ three times and introducing unknowns from the last column.

$\frac{b-22}{1} = 1 \Rightarrow b = 23,$ $\qquad \frac{p-228}{-2} = 23 \Rightarrow p = 182$

$$\frac{a-922}{-4} = 182 \Rightarrow a = 194$$

$\frac{d-b}{-2} = 1 \Rightarrow d = 21;$ $\quad \frac{q-p}{-4} = 21 \Rightarrow q = 98.$

$$\frac{k-d}{4} = 1 \Rightarrow k = 17.$$

Now take 5 as the origin and proceed.

$$f(x) = f(5) + (x-5)\, f(x_0, x_1) + (x-5)(x-5)\, f(x_0, x_1, x_2) + (x-5)^3 f(x_0, x_1, x_2, x_3)$$

$$= a + (x-5)q + (x-5)^2 k + (x-5)^3 \times 1$$

$$= 194 + 98(x-5) + 17(x-5)^2 + (x-5)^3$$

4. Find the function $f(x)$ from the following table hence evaluate $f(x)$.

x:	0	1	2	4	5	7
$f(x)$:	0	0	-12	0	600	7308

Sol: Since 6 data are given, we assume the polynomial to be of degree 5.

Since $f(0)=0$, $f(1)=0$ & $f(4)=0$, it is clear $x(x-1)(x-4)$ is a factor of $f(x)$,

So, let $f(x) = x(x-1)(x-4)\,\phi(x)$ where $\phi(x)$ is a quadratic polynomial.

Now, $\phi(x) = \dfrac{f(x)}{x(x-1)(x-4)}$

$\therefore \phi(2) = \dfrac{f(2)}{2(1)(-2)} = \dfrac{-12}{-4} = 3$

$\phi(5) = \dfrac{f(5)}{5(4)(1)} = \dfrac{600}{20} = 30$

$\phi(7) = \dfrac{f(7)}{7(6)(3)} = \dfrac{7308}{126} = 58$

Now we will find $\phi(x)$ using divided difference formula of Newton,

x	$\phi(x)$	$\Delta\phi(x)$	$\Delta^2\phi(x)$
2	3		
		9	
5	30		1
		14	
7	58		

By Newton's formula,

$\phi(x) = \phi(x_0) + (x-x_0)\phi(x_0,x_1) + (x-x_0)(x-x_1)\phi(x_0,x_1,x_2)$

$= 3 + (x-2)9 + (x-2)(x-5)$

$= x^2 + 2x - 5$.

Hence $f(x) = x(x-1)(x-4)(x^2+2x-5)$

Cubic Spline interpolation

We define a Cubic Spline, $s(x)$ as follows.

(i) $s(x)$ is a polynomial of degree one for $x<x_0$ and $x>x_n$.

(ii) $s(x)$ is at most a Cubic polynomial in each interval (x_{i-1}, x_i), $i=1,2,\ldots n$

(iii) $s(x)$, $s'(x)$ and $s''(x)$ are continuous at each point (x_i, y_i), $i=0,1,2\ldots n$ and

(iv) $s(x_i) = y_i$, $i=0,1,2\ldots n$

Method 1:

For Convenience, we assume equal interval (ie) $x_i - x_{i-1} = h$, $i=1,2,3\ldots n$. Since there are n equal intervals, we have to find n cubic polynomials totally. Hence, if the number of intervals is large, it is not easy to find all these polynomials - cubic splines.

Since $s(x)$ is a Cubic polynomial, $s''(x)$ is linear in each interval.

In the interval (x_{i-1}, x_i),

(x_0, x_1)

Let us assume

$$s''(x) = \frac{1}{h}\left[(x_i - x)\, s''(x_{i-1}) + (x - x_{i-1})\, s''(x_i)\right] \quad (1)$$

We can easily check that this equation is valid when we put $x = x_{i-1}$ & $x = x_i$.

Integrating twice

$$s(x) = \frac{1}{h}\left[\frac{(x_i-x)^3}{3!} s''(x_{i-1}) + \frac{(x-x_{i-1})^3}{3!} s''(x_i)\right]$$
$$+ a_i(x_i-x) + b_i(x-x_{i-1}) \longrightarrow (2)$$

Where a_i, b_i are constants to be found out by using the conditions,

$$s(x_i) = y_i \text{ (given)}, \quad i = 0,1,2 \ldots n$$

put $x = x_{i-1}$ in (2), we get

$$y_{i-1} = \frac{1}{h}\left[\frac{h^3}{3!} s''(x_{i-1})\right] + h a_i$$

$$\therefore a_i = \frac{1}{h}\left\{y_{i-1} - \frac{h^2}{3!} s''(x_{i-1})\right\}$$

put $x = x_i$ in (2), we get

$$b_i = \frac{1}{h}\left[y_i - \frac{h^2}{3!} s''(x_i)\right]$$

Hence the equation (2) reduces to

$$s(x) = \frac{1}{h}\left\{\frac{(x_i-x)^3}{3!} s''(x_{i-1}) + \frac{(x-x_{i-1})^3}{3!} s''(x_i)\right\}$$
$$+ \frac{1}{h}(x_i-x)\left\{y_{i-1} - \frac{h^2}{3!} s''(x_{i-1})\right\}$$
$$+ \frac{1}{h}(x-x_{i-1})\left\{y_i - \frac{h^2}{3!} s''(x_i)\right\} \longrightarrow (3)$$

Writing $s''(x_i) = M_i$, the above equation becomes

$$s(x) = \frac{1}{6h}\left[(x_i-x)^3 M_{i-1} + (x-x_{i-1})^3 M_i\right]$$
$$+ \frac{1}{h}(x_i-x)\left[y_{i-1} - \frac{h^2}{6} M_{i-1}\right] + \frac{1}{h}(x-x_{i-1})\left[y_i - \frac{h^2}{6} M\right.$$
$$\longrightarrow (4$$

The quantities M_i which are the spline second derivatives are not yet known.
Now we will impose the continuity of $s'(x)$

from ④ $s'(x) = \frac{1}{6h}\left[3(x_i - x)^2(-M_{i-1}) + 3(x - x_{i-1})^2 M_i\right]$

$$+ \frac{1}{h}\left[-y_{i-1} + \frac{h^2}{6}M_{i-1}\right] + \frac{1}{h}\left[y_i - \frac{h^2}{6}M_i\right]$$

$$\therefore s'(x_{i-}) = \frac{h}{3}M_i + \frac{h}{6}M_{i-1} + \frac{1}{h}(y_i - y_{i-1}) \quad \text{⑤}$$

Illy $s'(x_{i+}) = -\frac{h}{3}M_i - \frac{h}{6}M_{i+1} + \frac{1}{h}(y_{i+1} - y_i) \quad$ ⑥

Equating ⑤ & ⑥, we get

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right]$$
for $i = 1, 2, 3 \ldots (n-1)$ ⑦

Further, in view of the first condition, that $s(x)$ is linear for $x < x_0$ & $x > x_n$.
We have $s''(x) = 0$ at $x = x_0$ & $x = x_n$.
Hence $M_0 = 0$, $M_n = 0$ ⑧

Equ. ⑦ & ⑧ give $(n+1)$ equ. in $(n+1)$ unknown $M_0, M_1, \ldots M_n$. Hence we can solve for $M_0, M_1, M_2 \ldots M$.

Substituting in ④, we get the cubic spline in each interval.

1. From the following table

x	1 (x_0)	2 (x_1)	3 (x_n)
y	-8	-1	18

Compute $y(1.5)$ & $y'(1)$. Using Cubic Spline.

Soln:

Here $h=1$, & $n=2$. Also assume $M_0=0$ & $M_2=0$

We have $M_{i-1}+4M_i+M_{i+1} = \frac{6}{h^2}[y_{i-1}-2y_i+y_{i+1}]$ for $i=1,2 \ldots (n-1)$

From this $i=1$

$$M_0+4M_1+M_2 = 6[y_0-2y_1+y_2]$$

$$4M_1 = 6[-8-2(-1)+18] = 72$$

$$M_1 = 18$$

W.K.T $S(x) = \frac{1}{6h}\left[(x_i-x)^3M_{i-1} + (x-x_{i-1})^3M_i\right] + \frac{1}{h}(x_i-x)\left[y_{i-1}-\frac{h^2}{6}M_{i-1}\right] + \frac{1}{h}(x-x_{i-1})\left[y_i-\frac{h^2}{6}M_i\right]$ — (1)

From (1), for $1 \le x \le 2$, putting $i=1$, we get

$$S(x) = \frac{1}{6h}\left[(x_1-x)^3M_0 + (x-x_0)^3M_1\right] + \frac{1}{h}(x_1-x)\left[y_0-\frac{h^2}{6}M_0\right] + \frac{1}{h}(x-x_0)\left[y_1-\frac{h^2}{6}M_1\right]$$

$$= \frac{1}{6}\left\{18(x-1)^3\right\} + (2-x)(-8) + [(-(x-1) - (x-1)^3]$$

$$= \frac{1}{6}\left\{18(x-1)^3\right\} + (2-x)(-8) - 4(x-1)$$

$$= 3(x-1)^3 + 4x - 12$$

$$= 3x^3 - 9x^2 + 13x - 15$$

$\Rightarrow y(1.5) = -\frac{45}{8}$ & $y' \approx S'(x) = 9(x-1)^2+4 \Rightarrow y'(1) = 4.$

2) Using Cubic Spline, find $y(0.5)$ & $y'(1)$ given $M_0 = M_2 = 0$ and the table.

x	0	1	2
y	-5	-4	3

Solu: Here $h=1$, & $n=2$

we have,

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\{y_{i-1} - 2y_i + y_{i+1}\}$$

for $i = 1, 2 \ldots n-1$

Putting $i=1$ $M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2]$

$$= 6[-5 - 2(-4) + 3] = 36$$

$$4M_1 = 36 \Rightarrow M_1 = 9.$$

We will derive the cubic spline in $[0,1]$

W.K.T $S(x) = \frac{1}{6h}\{(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\}$

$$+ \frac{1}{h}(x_i - x)\{y_{i-1} - \frac{h^2}{6}M_{i-1}\} + \frac{1}{h}(x - x_{i-1})\{y_0 - \frac{h^2}{6}M_i\}$$

for $i = 1, 2, 3 \ldots n$

put $S(x) = \frac{1}{6}\{(x_1 - x)^3 M_0 + (x - x_0)^3 M_1\} + \frac{1}{1}(x_1 - x)[y_0 - \frac{h^2}{6}M_0]$

$$+ \frac{1}{1}(x - x_0)[y_1 - \frac{1}{6}M_1)$$

$$= \frac{1}{6}\{(1-x)^3 0 + (x-0)^3 9]$$

$$+ [(1-x)][-5 - \frac{1}{6}(0)] + [(x-0)[-4 - 9/6]$$

$$= \frac{1}{6}[9x^3] - 5(1-x) - \frac{33}{6}x$$

$$= \frac{3}{2}x^3 - \frac{x}{2} - 5, \text{ where } 0 \le x \le 1.$$

$$S(0.5) = \frac{3}{2}(\tfrac{1}{8}) - \tfrac{1}{4} - 5 = -81/16$$

$$S'(x) = \frac{9}{2}x^2 - 1/2 \Rightarrow S'(1) \approx y'(1) = \frac{9}{2} - 1/2 = 4,$$

3. Find the Cubic Spline approximation for the function given below

x	0	1	2	3
y	1	2	33	244

Assume $M(0) = M(3) = 0$. Also find $y(2.5)$

Soln: Here $h=1$, & $n=3$.

We have $M_{i-1}+4M_i+M_{i+1} = \frac{6}{h^2}[y_{i-1}-2y_i+y_{i+1}]$ —— (1)

for $i=1,2$.

$\therefore M_0+4M_1+M_2 = 6[y_0-2y_1+y_2]$

$M_1+4M_2+M_3 = 6[y_1-2y_2+y_3]$

These reduces to, [taking $M_0=0$, $M_3=0$]

$4M_1+M_2 = 6(1-4+33) = 180$

$M_1+4M_2 = 6(2-66+244) = 1080$

Solving $M_1 = -24$, $M_2 = 276$.

W.k.T

$$S(x) = \frac{1}{6h}\{(x_i-x)^3 M_{i-1} + (x-x_{i-1})^3 M_i\} + \frac{1}{h}(x_i-x)\{y_{i-1} - \frac{h^2}{6}M_{i-1}\} + \frac{1}{h}(x-x_{i-1})\{y_i - \frac{h^2}{6}M_i\}$$

for $i=1,2,3\cdots n$ —— (2)

Put $i=1$ in equ. (2)

$$S(x) = \frac{1}{6h}\{(x_1-x)^3 M_0 + (x-x_0)^3 M_1\} + \frac{1}{h}(x_1-x)(y_0-\frac{h^2}{6}M_0) + \frac{1}{h}(x-x_0)(y_1-\frac{h^2}{6}M_1)$$

$$= \frac{1}{6}[0+(x-0)^3(-24)] + (1-x)[1-0] + (x-0)(2+\frac{24}{6})$$

$$= -4x^3+5x+1$$ —— (I)

In [1,2], [put $i=2$ in (2)]

$$S(x) = \frac{1}{6}\left[(2-x)^3(-24) + (x-1)^3(276)\right]$$
$$+(2-x)\left[2+\frac{24}{6}\right] + (x-1)\left(33-\frac{276}{6}\right]$$
$$= 50x^3 - 162x^2 + 167x - 53 \quad \text{——— II}$$

In [2,3] [put $i=3$ in (2)]

$$S(x) = \frac{1}{6}\left\{(3-x)^3 M_2\right\} + (3-x)\left(33-\frac{M_2}{6}\right) + (x-2)(244-0)$$

$$= \frac{1}{6}\left\{(3-x)^3(276)\right\} + (3-x)(33-46) + (x-2)(244)$$

$$= 46(3-x)^3 - 13(3-x) + 244(x-2)$$

$$= -46x^3 + 414x^2 - 985x + 715 \quad \text{——— III}$$

Equ. I, II & III give the cubic spline in each sub interval.

$y(2.5) = 121.25$ //

(4) Test whether the following func. are cubic spline (or) not

$P_1(x) = x^2 - x + 1, \quad 1 \le x \le 2$

$P_2(x) = 3x - 3, \quad 2 \le x \le 3$

Each poly. is at most of degree three in each sub-interval.

$P_1(2) = 3 = P_2(2)$

$P_1'(2) = 3 = P_2'(2)$

$P_1''(2) = 2, \quad P_2''(2) = 0$

$\therefore$ Not a cubic spline since $S''(x)$ is not continuous at $x=2$.

Newton forward and Backward difference formula

1. Derive Newton's forward difference formula by using operator method:

Solu: $p_n(x) = p_n(x_0+uh) = E^u p_n(x_0)$

$= E^u y_0 = (1+\Delta)^u y_0$

$= 1 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots$

$\dots \frac{u(u-1)(u-2)\dots(u-\overline{n-1})}{n!}\Delta^n y_0$

where $u = \frac{x-x_0}{h}$.

2. Derive Newton's backward difference formula by using operator method.

Solu: $p_n(x) = p_n(x_n+vh) = E^v p_n(x_n)$

$= (1-\nabla)^{-v} y_n$

$= [1 + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \dots$

$\dots \frac{v(v+1)(v+2)\dots(v+n-1)}{n!}\nabla^n y_n$

Where $v = \frac{x-x_n}{h}$

Problems:

1. From the following data, find θ at $x=43$ and $x=84$.

x :	40	50	60	70	80	90
θ :	184	204	226	250	276	304

Also express θ in terms of x.

1. ~~From the following data, find~~

Soln: Since Six data are given, p(x) is of degree 5. To find θ at x=43 use forward interpolation and to find θ at x=84, use backward interpolation formula.

$$u = \frac{x - x_0}{h} = \frac{43-40}{10} = 0.3$$

$$\& \; v = \frac{84-90}{10} = \frac{x - x_n}{h} = -0.6$$

Table

x	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$	$\Delta^5\theta$
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

Newton's forward formula is

$$\theta(x) = \theta_0 + \frac{u}{1!}\Delta\theta_0 + \frac{u(u-1)}{2!}\Delta^2\theta_0 + \ldots$$

$$= \theta[40 + (0.3)10]$$

$$\theta(x=43) = 184 + (0.3)20 + \frac{(0.3)(-0.7)(2)}{2}$$

$$= 189.79$$

Newton's Backward formula is

$$\theta(x) = \theta_n + v\nabla\theta_n + \frac{v(v+1)}{2}\nabla^2\theta_n + \dots$$

$$\theta(x=84) = \theta[90+(-0.6)10]$$

$$= 304(-0.6)28 + \frac{(-0.6)(0.4)}{2}(2) = 286.96$$

$$\theta = \theta_0 + u\Delta\theta_0 + \frac{u(u-1)}{2!}\Delta^2\theta_0 + \dots$$

where $u = \frac{x-40}{10}$

$$= 184 + u(20) + \frac{u(u-1)}{2!}(2)$$

$$= 184 + \frac{20(x-40)}{10} + \frac{(x-40)(x-50)}{100}$$

$$= 184 + 2x - 80 + \frac{1}{100}[x^2 - 90x + 2000]$$

$$= 0.01x^2 + 1.1x + 124 //$$

2) Find a polynomial of degree four which takes the Values

x :	2	4	6	8	10
y :	0	0	1	0	0

Sol: Let us form the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	0				
		0			
4	0		1		
		1		-3	
6	1		-2		6
		-1		3	
8	0		1		
		0			
10	0				

Let us find the polynomial using Newton's interpolation formula.

$$u = \frac{x - x_0}{h} = \frac{x-2}{2}$$

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 0 + u(0) + \frac{\left(\frac{x-2}{2}\right)\left(\frac{x-4}{2}\right)}{2}(1) + \frac{\left(\frac{x-2}{2}\right)\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)}{6}(-3)$$

$$+ \frac{\left(\frac{x-2}{2}\right)\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)\left(\frac{x-8}{2}\right)}{24}(6)$$

$$= \frac{(x-2)(x-4)}{8}\left[1 - \frac{1}{2}(x-6) + \frac{1}{8}(x-6)(x-8)\right]$$

$$= \frac{1}{64}\left[(x-2)(x-4)\right]\left[8 - 4x + 24 + x^2 - 14x + 48\right]$$

$$= \frac{1}{64}\left\{(x-2)(x-4)(x-8)(x-10)\right\}$$

$$= \frac{1}{64}\left\{x^4 - 24x^3 + 196x^2 - 624x + 640\right\}$$

3 Find a polynomial of degree two which takes the values

x :	0	1	2	3	4	5	6	7
y :	1	2	4	7	11	16	22	29.

4) Find the values of y at $x = 21$ & $x = 28$ from the following data.

x :	20	23	26	29
y :	0.3420	0.3907	0.4384	0.4848.

5) From the data given below, find the number of Students whose weight is between 60 & 70.

Weight :	0-40	40-60	60-80	80-100	100-120
No. of students :	250	120	100	70	50

Solu.

Diff. table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
Below 60	370		-20		
		100		-10	
					20
Below 80	470		-30		
		70		10	
Below 100	540		-20		
		50			
Below 120	590				

Let us Calculate the number of Students whose weight is less than 70.

We will use forward difference formula,

$$u = \frac{x - x_0}{h} = \frac{70-40}{20} = 1.5$$

$$y(70) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \dots$$

$$= 250 + (1.5)(120) + \frac{(1.5)(0.5)}{2}(-20)$$
$$+ \frac{(1.5)(0.5)(-0.5)}{6}(-10) + \frac{(1.5)(0.5)(-0.5)(-1.5)}{24}(20)$$

$$= 250 + 180 - 7.5 + 0.625 + 0.46875 \simeq 424.$$

$\therefore$ No. of Student whose weight is between 60 & 70

$$= y(70) - y(60) = 424 - 370 = 54$$ //

6) Find the missing value of the table given below.

x :	0	1	2	3	4
y :	1	2	4	-	16

Explain why $y(x=3)$ is not $2^3=8$ in your answer.

Soln: Since only 4 values are given, we assume the polynomial to be of third degree. Hence, fourth diff. of $P_3(x)$ are zero.

Taking $y_x = P_3(x)$, $y_0=1$, $y_1=2$, $y_2=4$, $y_4=16$.

Since $\Delta^4 y_0 = 0$, we have

$$(E-1)^4 y_0 = 0$$

$$(E^4-4E^3+6E^2-4E+1)y_0 = 0 \qquad \because Ef(x) = f(x+h)$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0 \qquad E^2f(x) = f(x+2h)$$

$$16 - 4y_3 + 6(4) - 4(2) + 1 = 0$$

$$\Rightarrow 4y_3 = 33 \qquad \therefore y_3 = 8.25$$

By looking at the table, we guess $y = 2^x$ is the function from which the given table is created. So, we should have got $y(3) = 2^3 = 8$. But our answer is 8.25, because in getting this answer, we have assumed a third degree polynomial. which is only an approximating polynomial and not the actual $y = 2^x$. Hence, the difference in the answer is got.

—— x —— x ——