UNIT - IV

Initial value problems for ordinary Differential Equations.

Single step (or) pointwise solution:

In solving a differential equation for approximate solution we find numerical values of $y_1, y_2, y_3 \ldots$ corresponding to given values of independent variable values $x_1, x_2, x_3 \ldots$ so that the ordered pairs (x_1, y_1) $(x_2, y_2) \ldots$ satisfy a particular solution, though approximately. A solution of this type is called a pointwise solution.

Single step methods (or) pointwise methods.

In these methods we use information about the curve at one point and we do not iterate the solution. The method involves more evaluation of the function, The methods of Taylor series, Euler & Runge-kutta belong to this type.

Multi step methods (or) step by step methods

These methods required fewer evaluation of the function (past four values of the function) to estimate the solution at a point and iterations are performed till sufficient accuracy is achieved. Estimation of error is possible and the methods are called Predictor - corrector methods.

The methods of Milne & Adam Bashforth belong to this type.

Taylor series method

Consider the 1st order differential equation

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0 \quad \text{——— (1)}$$

The solution of the above initial value problem is obtained in two types.

(i) power series solution &

(ii) pointwise solution.

(i) Power series solution:

If $y = y(x)$ is the solution of (1), then $y(x)$ can be expanded in a Taylor series about the point $x = x_0$, as

$$y(x) = y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!}\,y''(x_0) + \cdots$$

$$\Rightarrow y(x) = y_0 + \frac{(x-x_0)}{1!}\,y_0' + \frac{(x-x_0)^2}{2!}\,y_0'' + \cdots$$

where $y_0^{(r)} = \frac{d^r y}{dx^r}$ at (x_0, y_0)

Using (1), the derivatives $y_0', y_0'', y_0''' \cdots$ can be found by means of successive differentiation. Expressions (2) gives the values of y for every value of x for which (2) converges.

(ii) pointwise solution.

If $y(x)$ is the solution of (1), then by

Taylor series

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

Put $x_1 = x_0 + h$

$$y(x_1) = y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \quad (1)$$

Once y_1 has been calculated from (1),
y_1', y_1'', y_1''' can be calculated from
$$y' = f(x, y)$$

Expanding $y(x)$ in a Taylor series about $x = x_1$, we get $y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_1''' + \cdots$

where $y_2 = y(x_2)$ & $x_2 = x_1 + h$

The Taylor algorithm is given as follows

$$y_{m+1} = y_m + \frac{h}{1!}y_m' + \frac{h^2}{2!}y_m'' + \frac{h^3}{3!}y_m''' + \cdots$$

where $y_m^{(r)} = \frac{d^r y}{dx^r}$ at the pt. (x_m, y_m), where $m = 0, 1, 2 \ldots$

Eg 1) Solve $\frac{dy}{dx} = x + y$, given $y(1) = 0$, and get $y(1.1)$, $y(1.2)$ by Taylor series method. Compare your result with the explicit solution.

Soln:

Here $x_0 = 1$, $y_0 = 0$, $h = 0.1$, $y_0 = y(x=1) = 0$.

$y' = (x+y)$	$y_0' = x_0 + y_0 = 1 + 0 = 1$
$y'' = 1 + y'$	$y_0'' = 1 + y_0' = 2$
$y''' = y''$	$y_0''' + y_0'' = 2$
$y^{iv} = y'''$	$y_0^{iv} = y_0''' = 2$ etc

By Taylor series, we have

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots \quad \text{—— (1)}$$

$$\therefore y_1 = y(1.1) = 0 + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(0) + \frac{(0.1)^3}{6}(2) + \cdots \quad \text{—— (2)}$$

$$= 0.1 + 0.01 + 0.0033 + 0.00000833 + 0.000000166 + \cdots$$

$$y(1.1) = 0.11033847.$$

Now, take $x_0 = 1.1$, $h = 0.1$,

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \cdots \quad \text{—— (3)}$$

We calculate $y_1', y_1'', y_1''', \ldots$ $x_1 = 1.1$, $y_1 = 0.11033847$

$$y_1' = x_1 + y_1 = 1.1 + 0.11033847$$

$$= 1.21033847.$$

$$y_1'' = 1 + y_1' = 2.21033847.$$

$$y_1''' = y_1'' = y_1^{iv} = y_1^{v} = \cdots = 2.21033847.$$

Using in (3)

$$y_2 = y(1.2) = 0.11033847 + \frac{0.1}{1!}(1.21033847) + \frac{(0.1)^2}{2}(2.210338\ldots$$

$$+ \frac{(0.1)^3}{6}(2.21033847) + \frac{(0.1)^4}{24}(2.21033847) + \cdots$$

$$= 0.11033847 + 0.121033847 + 2.21033847 + (0.05 + 0.00166\ldots$$

$$= 0.24280160$$

The exact solu. of $\frac{dy}{dx} = x + y$ is $y = -x - 1 + 2e^{x-1}$

$$y(1.1) = -1.1 - 1 + 2e^{0.1} = 0.11034$$

$$y(1.2) = -1.2 - 1 + 2e^{0.2} = 0.2128.$$

2) Using Taylor series method, find the value of $y(0.1)$, correct to four decimal places, given $\frac{dy}{dx} = x^2+y^2$ & $y(0) = 1$.

Soln:

Given $y' = x^2+y^2$

$y'' = 2x + 2yy'$

$y''' = 2 + 2yy'' + 2y'^2$

$y^{iv} = 2yy''' + 2y'y'' + 4y'y''$

$= 2yy''' + 6y'y''$

$x_0 = 0,\ y_0 = 1,\ h = 0.1$

$x_1 = 0.1,\ y_1 = y(0.2) = ?$

$y_0' = x_0^2 + y_0^2 = 0+1 = 1$

$y_0'' = 2,\ y_0''' = 8$

$y_0^{iv} = 28.$

By Taylor series method

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$y(0.1) = 1 + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(8) + \frac{(0.1)^4}{24}(28) + \cdots$$

$= 1 + 0.1 + 0.01 + 0.001333 + 0.00011 6166$

$= 1.11144999$

$= 1.11145$ //

3. Using Taylor series method, find y at $x = 0.1(0.1)0.4$ given $\frac{dy}{dx} = x^2 - y,\ y(0) = 1$ (correct to 4 decimal places)

Soln.

$x_0 = 0,\ y_0 = 1,\ h = 0.1,\ x_1 = 0.1,\ x_2 = 0.2 \ldots$

$y' = x^2 - y$

$y'' = 2x - y'$

$y''' = 2 - y''$

$y^{iv} = -y'''$

$y_0' = x_0^2 - y_0 = -1$

$y_0'' = 0 - (-1) = 1$

$y_0''' = 2 - 1 = 1$

$y_0^{iv} = -1$ etc.

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{6} y_0''' + \frac{h^4}{24} y_0^{iv} + \cdots$$

$$y_1 = y(0.1) = 1 + (0.1)(-1) + \frac{(0.01)}{2}(1) + \frac{0.001}{6}(1) + \left(\frac{0.0001}{24}\right)(-1) + \cdots$$

$$= 1 - 0.1 + 0.005 + 0.0001666 - 0.000004167 \cdots$$

$$= 0.905125$$

$$y_1' = x_1^2 - y_1 = 0.01 - 0.905125$$

$$= -0.895125$$

$$y_1'' = 2x_1 - y_1' = 0.2 + 0.895125$$

$$= 1.095125$$

$$y_1''' = 2 - y_1' = 2 - 1.095125 = 0.904875$$

$$\therefore y_2 = y_1 + \frac{h}{1} y_1' + \frac{h^2}{2!} y_1'' + \cdots$$

$$y(0.2) = y_2 = 0.905125 + (0.1)(-0.895125) + \frac{0.01}{2}(1.095125)$$

$$+ \frac{0.001}{6}(0.904875) + \cdots$$

$$= 0.128268$$

Hly $y(0.3) = 0.7492$

$y(0.4) = 0.6897$.

Taylor series method for Simultaneous first order differential equation:

We can solve the equations of the form $\frac{dy}{dx} = f_1(x,y,z)$; $\frac{dz}{dx} = f_2(x,y,z)$ with initial conditions $y(x_0) = y_0$ & $z(x_0) = z_0$

The values of y & z at $x_1 = x_0 + h$ are given by Taylor algortham,

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

$$z_1 = z_0 + \frac{h}{1!} z_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \ldots$$

The deriviations on R.H.S of the above expressions are found at $x = x_0$ using given eqn.

Similarly y_2 & z_2 corresponding to $x_2 = x_1 + h$ are calculated by taylor series.

Ex 1) Solve $\frac{dy}{dx} = z - x$, $\frac{dz}{dx} = y + x$, with $y(0) = 1$, $z(0) = 1$ by taking $h = 0.1$, to get $y(0.1)$ & $z(0.1)$. Here y & z are dependent variables and x is independent

Soln

Take $x_0 = 0$, $y_0 = 1$

$y' = z - x$

$y'' = z' - 1$

$y''' = z''$ etc.

Take $x_0 = 0$, $z_0 = 1$ & $h = 0.1$

$z' = x + y$

$z'' = 1 + y'$

$z''' = y''$ etc.

$y_0 = 1$	$z_0 = 1$
$y_0' = z_0 - x_0 = 1 - 0 = 1$	$z_0' = x_0 + y_0 + 0 = 1$
$y_0'' = z_0' - 1 = 0$	$z_0'' = 1 + y_0' = 2$
$y_0''' = z_0'' = 2$	$z_0''' = y_0'' = 0$.

By Taylor series for y_1 & z_1, we have

$$y_1 = y(0.1) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \quad \text{(1)}$$

$$\& \; z_1 = z(0.1) = z_0 + h z_0' + \frac{h^2}{2} z_0'' + \frac{h^3}{6} z_0''' + \cdots \quad \text{(2)}$$

Substituting in (1) & (2), we get

$$y_1 = y(0.1) = 1 + (0.1) + \frac{0.01}{2}(0) + \frac{0.001}{6}(2) + \cdots$$

$$= 1 + 0.1 + 0.000333 + \cdots$$

$$= 1.1003$$

$$z_1 = z(0.1) = 1 + (0.1)(1) + \left(\frac{0.01}{2}\right)(2) + \frac{0.001}{6}(0) + \frac{0.0001}{24} \times 2 + \cdots$$

$$= 1 + 0.1 + 0.01 + 0.0000083 + \cdots$$

$$= 1.1100$$

$\therefore y(0.1) = 1.1003$

& $z(0.1) = 1.1100$.

2) Evaluate $x(0.1)$ & $y(0.1)$, $x(0.2)$, $y(0.2)$ given $\frac{dx}{dt} = 1 + ty$; $\frac{dy}{dt} = -tx$ given $x = 0$, $y = 1$ at $t = 0$ by Taylor series method.

Soln:

Given $t_0 = 0$, $x_0 = 0$, $y_0 = 1$

$x' = 1 + ty$	$y' = -tx$
$x'' = y + ty'$	$y'' = -[x + tx']$
$x''' = 2y' + ty''$	$y''' = -[2x' + tx'']$
$x^{iv} = 3y'' + ty'''$	$y^{iv} = -[3x'' + tx''']$

Then

$x_0' = 1$	$y_0' = 0$
$x_0'' = 1$	$y_0'' = 0$
$x_0''' = 0$	$y_0''' = -2$
$x_0^{iv} = 0$	$y_0^{iv} = -3$

By Taylor algorithm, we have

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' \ldots$$

$$\therefore y(0.1) = 1 + (0.1)(0) + \frac{0.01}{2}(0) + \frac{0.001}{6}(-2) + \frac{0.0001}{24}(-3) + \ldots$$

$$= 1 - 0.0003$$

$$y(0.1) = 0.9997.$$

$$x(0.1) = x_1 = x_0 + \frac{h}{1!} x_0' + \frac{h^2}{2!} x_0'' + \frac{h^3}{3!} x_0''' + \ldots$$

$$= 0 + (0.1)(1) + \left(\frac{0.01}{2}\right)(1) + \ldots$$

$$= 0.1 + 0.005$$

$$= 0.105,$$

Find $x_1', x_1'', x_1''', \ldots$ & $y_1', y_1'', \ldots$

Illy $x(0.2) = 0.21998.$

$y(0.2) = 0.9972$ //

3) By means of Taylor series expansion, find y at $x = 0.1, 0.2$ correct to three digits given $\frac{dy}{dx} - 2y = 3e^x$, $y(0) = 0$. Compare your result with exact solution.

Soln:

Here $x_0 = 0$, $y_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$ & $h = 0.1$

$y' = 2y + 3e^x$ | $y_0' = 2y_0 + 3e^{x_0} = 3$

$y'' = 2y' + 3e^x$ | $y_0'' = 2y_0' + 3e^{x_0} = 9$

$y''' = 2y'' + 3e^x$ | $y_0''' = 18 + 3 = 21$

$y^{iv} = 2y''' + 3e^x$ | $y_0^{iv} = 42 + 3 = 45$ //

$\therefore y_1 = y_0 + \frac{h}{1} y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0''' + \frac{h^4}{24} y^{iv} + \cdots$

$y(0.1) = y_1 = 0 + (0.1)3 + \left(\frac{0.01}{2}\right) 9 + \frac{0.001}{6}(21) + \frac{(0.01)}{24}(45) + \cdots$

$= 0.3 + 0.045 + 0.035 + 0.0001875 + \cdots$

$\approx 0.3486875 \approx 0.349$ //

$y_1' = 2y_1 + 3e^{x_1} = 4.012887$

$y_1'' = 2y_1' + 3e^{x_1} = 11.025774$

$y_1''' = 2y_1'' + 3e^{x_1} = 25.3670608$

$y_2 = y(0.2) = y_1 + \frac{h}{1} y_1' + \frac{h^2}{2} y_1'' + \cdots$

$= 0.3486875 + (0.1)(4.012887) + \frac{(0.01)}{2}(11.341286$

$+ \frac{(0.001)}{6}(25.99808) + \cdots$

$= 0.8110156$

≈ 0.811

The exact solu. of $\frac{dy}{dx} - 2y = 3e^x$,

$$p = -2 \quad , \quad Q = 3e^x$$

$$e^{-2\int dx} = e^{-2x}$$

$$ye^{\int p dx} = \int Q e^{\int p dx} dx + c.$$

$$ye^{-2x} = \int 3e^x . e^{-2x} dx + c.$$

$$= \int 3e^{-x} dx + c.$$

$$= -3e^{-x} + c.$$

$$\Rightarrow c = ye^{-2x} + 3e^{-x}.$$

Sub. $y(0) = 0$, (i.e) $x = 0 \Rightarrow y = 1$

$$c = 0 + 3\cancel{e^{-1}} \Rightarrow c = 3\cancel{e^{-1}}$$

$$ye^{-2x} = -3e^{-x} + 3$$

(ie) $y = -3e^x + 3e^{2x}$

$$y(0.1) = 0.3486955$$

$$y(0.2) = 0.8412658 \; //$$

Euler's method

In Taylor series method, we obtain approximate solutions of the initial value problem $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ as a power series in x, & the solution can be used to compute y numerically specified values x near x_0.

In Euler methods, we compute the values of y for $x_i = x_0 + ih$; $i = 1, 2 \ldots$ with a step size $h>0$.

(ie) $y_i = y(x_i)$, where $x_i = x_0 + ih$, $i = 1, 2, 3 \ldots$

Euler Method:

Let $y_i = y(x_i)$, where $x_1 = x_0 + h$

Then $y_1 = y(x_0+h)$. Then by Taylor series,

$$y_1 = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \ldots \quad \text{——} \ (1)$$

Neglecting the terms with h^2 & higher powers of h, we get from (1)

$$y_1 = y_0 + h f(x_0, y_0) \quad \text{——} \ (2)$$

Expression (2) gives an approximate value of y at $x_1 = x_0 + h$.

Ily, we get $y_2 = y_1 + h f(x_1, y_1)$ for $x_2 = x_1 + h$.

$\therefore$ For any n,

$$y_{n+1} = y_n + h f(x_n, y_n), \ n = 0, 1, 2 \ldots$$

$$\text{(or)} \quad y(x+h) = y(x) + h f(x,y) \quad \text{——} \ (3)$$

This formula is called Euler's algorithm Error $= O(h^2)$.

Modified Euler's method:

$$y_{n+1} = y_n + h\left[f\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}h f(x_n, y_n)\right)\right] \quad \text{--- } (4)$$

(or) $$y(x+h) = y(x) + h\left[f\left(x + \tfrac{1}{2}h,\ y + \tfrac{1}{2}h f(x,y)\right)\right] \quad \text{--- } (5)$$

Equn. (4) (or) (5) called as modified Euler's formula.

Note:

In Euler method $y_{n+1} = y_n + \Delta y$

Where $\Delta y = hf(x_0, y_0)$ where $f(x_0, y_0)$ = slope at (x_0, y_0)

In modified E.M $y_{n+1} = y_n + \Delta y$.

Where $\Delta y = h$[average of the slopes at x_0 & x_1]

(or)

$= h$[average of the values of $\frac{dy}{dx}$ at the end of the interval x_0 to x_1]

1. Using Euler's method, solve numerically the equation $y' = x + y$, $y(0) = 1$, for $x = 0.0(0.2)(1.0)$. Check your answer with the exact solution.

Soln:

$f(x,y) = x + y$, $y_0 = 1$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$

$x_4 = 0.8$, $x_5 = 1.0$

By Euler algorithm,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= y_0 + h[x_0 + y_0]$$

$$= 1 + (0.2)(0+1)$$

(ie) $y(0.2) = 1.2$

$y_2 = y_1 + h f(x_1, y_1)$

$= 1.2 + (0.2)[x_1 + y_1]$

$= 1.2 + (0.2)[0.2 + 1.2]$

$= 1.2 + 0.28$

$y(0.4) = 1.48$

$y_3 = y_2 + h f[x_2, y_2]$

$= 1.48 + (0.2)[x_2 + y_2]$

$= 1.48 + (0.2)[0.4 + 1.48]$

$= 1.48 + 0.376$

$y(0.6) = 1.856$

$y_4 = 1.856 + 0.2(0.6 + 1.856) = 2.3472$

$y_5 = 2.3472 + (0.2)(0.8 + 2.3472)$

$= 2.94664$ //

To find exact soln:

Given $\frac{dy}{dx} - y = x$

$e^{\int P dx} = e^{-x}$, $Q = x$

$\therefore y e^{\int P dx} = \int Q e^{\int P dx} dx + c.$

$y e^{-x} = \int x e^{-x} dx + c. \Rightarrow C = -x - 1 + ce^{x}$

Given $y(0) = 1 \Rightarrow c = 2$

$\therefore y = 2e^{x} - x - 1$

x	0	0.2	0.4	0.6	0.8	1.0
Euler y	1	1.2	1.48	1.856	2.3472	2.94664
Exact y	1	1.2428	1.5836	2.0442	2.6511	3.4366

2. Using E.M find the solu. of the initial value problem $\frac{dy}{dx} = \log(x+y)$, $y(0) = 2$ at $x = 0.2$ by assuming $h = 0.2$.

Solu:

Given $f(x,y) = \log(x+y)$, $x_0 = 0$, $y_0 = 2$, $x_1 = 0.2$

$h = 0.2$.

By Euler's Algorithm,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= y_0 + h[\log(x_0 + y_0)]$$

$$= 2 + (0.2)\log(0+2)$$

$$= 2 + 0.2 \log 2$$

$$= 2 + (0.2)(0.3010)$$

$$= 2.0602 //$$

(ie) $y(0.2) = 2.0602$ //

Problems - Modified Euler's method.

1. Solve the equation $\frac{dy}{dx} = 1 - y$, given $y(0) = 0$ using modified Eulers method and tabulate the solutions at $x = 0.1, 0.2, \& 0.3$. Compare your result with the exact solution.

Solu:

Given $f(x,y) = 1 - y$, $x_0 = 0$, $y_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $h = 0.1$

By Modified Euler method

$y_{n+1} = y_n + h f[x_n + h/2, \; y_n + \frac{1}{2} h f(x_n, y_n)]$ —— (1)

$y_1 = y_0 + h f[x_0 + h/2, \; y_0 + h/2 \, f(x_0, y_0)]$ —— (2)

$f(x_0, y_0) = 1 - y_0 = 1 - 0 = 1$

(2) $\Rightarrow$ $y_1 = 0 + (0.1) f[0 + \frac{0.1}{2}, \; 0 + \frac{0.1}{2}(1)]$

$= (0.1) f[0.05, 0.05)$

$= (0.1) \; (1 - 0.05)$

$= (0.1) \; (0.95)$

$= 0.095$

$y_2 = y_1 + h f[x_1 + h/2, \; y_1 + h/2 \, f(x_1, y_1)]$ —— (3)

$f(x_1, y_1) = 1 - y_1$

$= 1 - 0.095$

$= 0.905$

(3) $\Rightarrow y_2 = (0.095) + (0.1) f[(0.1) + \frac{0.1}{2}, (0.095) + \frac{0.1}{2}(0.905)]$

$= 0.095 + (0.1) f(0.15, 0.14025)$

$= 0.095 + 0.1 (1 - 0.14025)$

$= 0.095 + 0.085975$

$= 0.18098.$

$y_3 = y_2 + h f[x_2 + h/2, \; y_2 + h/2 \, f(x_2, y_2)]$ —— (4)

$f(x_2, y_2) = 1 - y_2 = 1 - 0.18098$

$= 0.81902$

$= 0.18098 + (0.1) f[0.2 + \frac{0.1}{2}, \; 0.18098 + \frac{0.1}{2}(0.81902)]$

$= 0.18098 + (0.1) f[(0.25, \; 0.18098 + 0.04095)]$

$= 0.18098 + (0.1)[1 - 0.221931]$

$\Rightarrow (0.18098) + 0.0778069$

$= 0.258787$

$y(0.1) = 0.095$

$y(0.2) = 0.18098, \quad y(0.3) = 0.258787$

Exact solu:

$$\frac{dy}{dx} = 1-y, \text{ given } \frac{dy}{1-y} = dx$$

$$\therefore -\log(1-y) = x + c$$

$$\log(1-y) = -x - c$$

$$\therefore 1-y = e^{-x}.A$$

At $x=0, y=0 \Rightarrow A=1$

$$\therefore y = 1 - e^{-x} \;//$$

Using this exact solu.

$y(0.1) = 1 - e^{-0.1} = 0.9516258$

$y(0.2) = 1 - e^{-0.2} = 0.181269247$

$y(0.3) = 1 - e^{-0.3} = 0.25181779$

The values are tabulated,

x	M.E	Exact solu.
0.1	0.95	0.9516
0.2	0.18098	0.18127
0.3	0.258787	0.25918

2) Using Modified Euler method, find $y(0.2)$, $y(0.1)$ given $\frac{dy}{dx} = x^2+y^2$, $y(0) = 1$.

Solu:

Here $x_0 = 0$, $y_0 = 1$, $h = 0.1$, $x_1 = 0.1$, $x_2 = 0.2$,

$f(x,y) = x^2+y^2$

By Modified Euler method

$y_1 = y_0 + hf\left[x_0 + h/2,\ y_0 + h/2\, f(x_0, y_0)\right]$ — (1)

$\left[y_0 + h/2\, f(x_0, y_0)\right.$

$= y_0 + \frac{1}{2} h (x_0^2 + y_0^2)$

$= 1 + \frac{0.1}{2}(0+1) = 1.05$ // $\left.\right]$

Using in (1)

$y_1 = 1 + (0.1)\, f(0.05, 1.05)$

$y(0.1) = 1 + 0.1\left[(0.05)^2 + (1.05)^2\right]$

$= 1.1105$

$y_2 = y_1 + hf\left[x_1 + h/2,\ y_1 + h/2\, f(x_1, y_1)\right]$

$f(x_1, y_1) = f(0.1, 1.1105)$

$= (0.1)^2 + (1.1105)^2 = 1.243$ //

$y_1 + \frac{1}{2} h f(x_1, y_1) = 1.1105 + (0.05)(1.24321)$

$= 1.172160$

$\therefore y_2 = 1.1105 + (0.1)\, f(0.15, 1.172660)$

$= 1.1105 + (0.1)\left[(0.15)^2 + (1.17266)^2\right]$

$y(0.2) = 1.25026$ //

Runge-kutta Methods

R.K methods for solving 1st order equ:

2nd order R.K Method

If the initial values of (x,y) for the diff. Equation $\frac{dy}{dx} = f(x,y)$, then the 1st increment in y namely Δy is calculated from the formula.

$$k_1 = hf(x,y)$$

$$k_2 = hf(x+h/2,\ y+k_1/2)$$

$$\Delta y = k_2 \quad \text{where } h = \Delta x$$

$$\therefore y(x+h) = y(x) + \Delta y$$

Third order R.k. method:

The algorithm for this method is given below.

$$k_1 = hf(x,y)$$

$$k_2 = hf(x+h/2,\ y+k_1/2)$$

$$k_3 = hf[x+h,\ y+2k_2-k_1]$$

$$\& \quad \Delta y = \frac{1}{6}[k_1+4k_2+k_3]$$

$$\therefore y(x+h) = y(x) + \Delta y.$$

Fourth order R.k method for solving 1st order equ.

The algorithm for this method is given below

$$k_1 = hf(x,y)$$

$$k_2 = hf[x+h/2,\ y+k_1/2]$$

$$k_3 = hf[x+h/2,\ y+k_2/2]$$

$$k_4 = hf[x+h,\ y+k_3] \quad \& \quad \Delta y = \frac{1}{6}[k_1+2k_2+2k_3+k_4]$$

$$\therefore y(x+h) = y(x) + \Delta y.$$

1. Apply the fourth order R.k method to find $y(0.2)$ given that $y' = x+y$, $y(0) = 1$.

Solu:

Since h is not mentioned in the question, we take $h = 0.1$

Given $y' = x+y$; $y(0) = 1$

$\therefore f(x,y) = x+y$, $x_0 = 0$, $y_0 = 1$

$x_1 = 0.1$, $x_2 = 0.2$

By fourth order R.k method, for the 1st interval.

$K_1 = hf(x_0, y_0)$

$= (0.1)(x_0 + y_0)$

$= (0.1)(0+1)$

$= 0.1$ //

$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$

$= (0.1) f(0.05, 1.05)$

$= (0.1)(0.05 + 1.05) = 0.11$

$k_3 = hf(x_0 + h/2, y_0 + k_2/2)$

$= (0.1)(0.05, 1.055)$

$= (0.1)(0.05 + 1.055) = 0.1105$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= (0.1) f(0.1, 1.1105)$

$= (0.1)(0.1 + 1.1105)$

$= 0.12105$ //

$\therefore \Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105)$

$= 0.11034 1667$

$y(0.1) = y_1 = y_0 + \Delta y \approx 1.110342$ //

Now starting from (x_1, y_1) we get (x_2, y_2)

Again apply R.k algorithm replacing (x_0, y_0) by (x_1, y_1)

$k_1 = hf(x_1, y_1)$

$= (0.1)(x_1 + y_1)$

$= (0.1)(0.1 + 1.110342)$

$= 0.1210342$ //

$k_2 = hf(x_1 + h/2, y_1 + k_1/2)$

$= (0.1)(0.15 + 1.1708590)$

$= 0.1320859$ //

$k_3 = hf(x_1 + h/2, y_1 + k_2/2)$

$= (0.1)(0.15, 1.17763848)$

$(0.1)(0.15 + 1.17763848)$

$= 0.13263848$

$k_4 = hf(x_1 + h, y_1 + k_3)$

$= (0.1) f(0.2, 1.24298048)$

$= 0.144298048$

$\Delta y = \frac{1}{6}(k_1 + 2k_2) + 2k_3 + k_4) = \frac{1}{6}(0.79478 1008)$

$y(0.2) = y_1 + \Delta y = 1.110342 + \frac{1}{6}(0.79478 1008)$

$y(0.2) \approx 1.2428$ //

Multistep Methods:

These methods required fewer evaluation of the function (pass four values of the function) to estimate the solution at a point and iterations are performed till sufficient accuracy is achieved. Estimation of error is possible & the methods are called predictor. Corrector methods. The methods of Milne & Adam Bash forth belong to this type.

Milne's predictor & Corrector Method.

1. Write Milne's predictor corrector formula.

Milne's predictor formula is

$$y_{n+1,p} = y_{n-3} + \frac{4}{3}h\,(2y'_{n-2} - y'_{n-1} + 2y'_n)$$

Milne's corrector formula is

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}(y'_{n-1} - 4y'_n + y'_{n+1})$$

Note:

A predictor formula is used to predict the value of y at x_{i+1} & A corrector formula is used to correct the error & to improve that value of y_{i+1}.

1) Using Milne's method find $y(4.4)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ & $y(4.3) = 1.0143$.

Soln:

$$y' = \frac{2-y^2}{5x}, \quad x_0 = 4, \quad x_1 = 4.1, \quad x_2 = 4.2, \quad x_3 = 4.3$$

$$x_4 = 4.4, \quad y_0 = 1, \quad y_1 = 1.0049, \quad y_2 = 1.0097, \quad y_3 = 1.0143.$$

$$y_1' = \frac{2-y_1^2}{5x_1} = \frac{2-(1.0049)^2}{5(4.1)} = 0.0493.$$

$$y_2' = \frac{2-y_2^2}{5x_2} = \frac{2-(1.0097)^2}{5(4.2)} = 0.0467.$$

$$y_3' = \frac{2-y_3^2}{5x_3} = \frac{2-(1.0143)^2}{5(4.3)} = 0.0452.$$

By Milne's predictor formula

$$y_{4,p} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$= 1 + 4\frac{(0.1)}{3}[2(0.0493) - 0.0467 + 2(0.0452)]$$

$$= 1.01897$$

$$y_4' = \frac{2-y_4^2}{5(x_4)} = \frac{2-(1.01897)^2}{5(4.4)} = 0.0437$$

Using $y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$

$$= 1.0097 + \frac{0.1}{3}[(0.0467) + 4(0.0452) + 0.0437]$$

$$y_{4,c} = 1.01874 \;//$$

2. Given $y' = 1-y$ & $y(0)=0$, find
(i) $y(0.1)$ by Euler method.
(ii) $y(0.2)$ by Modified Euler method.
(iii) $y(0.4)$ by Milne's method.

Soln:

By Euler method,

$$y_1 = y_0 + hf(x_0, y_0) = 0 + (0.1)(1-y_0) = 0.1$$

By Modified Euler method.

$$y_2 = y_1 + hf\left[x_1 + \tfrac{1}{2}h,\ y_1 + \tfrac{1}{2}hf(x_1, y_1)\right]$$

$$= (0.1) + (0.1) f\left[0.1 + \frac{0.1}{2},\ 0.1 + \tfrac{1}{2}(0.1) f(0.1, 0.1)\right]$$

$$= (0.1) + (0.1)\left[1 - \left(0.1 + \frac{0.1}{2}(1-0.1)\right)\right]$$

$$= 0.1855$$

Ily $y_3 = 0.2629$.

Now knowing y_0, y_1, y_2, y_3 we will find

By Milne's method.

$$y_{4,P} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 0 + \frac{4(0.1)}{3}\left[2(1-y_1) - (1-y_2) + 2(1-y_3)\right]$$

$$= \frac{0.4}{3}\left[3 - 2y_1 + y_2 - 2y_3\right] = 0.3280$$ //

$$y_4' = 1 - y_4 = 1 - 0.3280 = 0.6720.$$

$$y_{4,C} = y_2 + h/3\,(y_2' + 4y_3' + y_{4,P}')$$

$$= 0.1855 + \frac{0.1}{3}\left[1 - y_2 + 4(1-y_3) + (1 - y_{4,P})\right)$$

$$= 0.1855 + \frac{0.1}{3}\left(6 - y_2 - 4y_3 - y_{4,P}\right]$$

$y(0.4) = 0.3393$ //

Adams-Bashforth predictor & corrector Method.

1. Write down Adams-Bashforth predictor & corrector formulae.

Adam's predictor & corrector formulae are

$$y_{n+1,p} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}]$$

& $$y_{n+1,c} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}]$$

2. How many prior values are required to predict the next value in Adam's method?

Four prior values.

3. What is a predictor-corrector method of solving a differential equation.?

predictor-corrector methods are methods which require the values of y at $x_n, x_{n-1}, x_{n-2}, x_{n-3}$ for for computing the values of y at x_{n+1}. we first use a formula to find the value of y at x_{n+1} and this is known as predictor formula. The value of y so got is corrected by another formula known as corrector formula.

1) Find $y(0.1), y(0.2), y(0.3)$ from $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$ by using RK method & hence obtain $y(0.4)$ using Adam's method.

Solu:

$f(x,y) = xy + y^2$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$

$k_1 = hf(x_0, y_0)$

$= (0.1) f(0,1) = (0.1) 1 = 0.1$

$k_2 = hf(0.05, y_0 + k_1/2)$

$= (0.1) f(0.05, 1.05)$

$= (0.1)[(0.05)(1.05) + (1.05)^2] = 0.1155$

$K_3 = hf(0.05, y_0 + k_2/2)$

$= (0.1) f(0.05, 1.0578)$

$= (0.1)[0.05(1.0578) + (1.0578)^2]$

$= 0.1172$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= (0.1) f(0.1, 1.1172)$

$= (0.1)[(0.1)(1.1172) + (1.1172)^2]$

$= 0.13598$

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$y(0.1) = 1.1169$

Again, ~~Strat~~ Start from y_1

$k_1 = hf(x_1, y_1) = (0.1) f(0.1, 1.1169)$

$= 0.1359$.

$k_2 = hf[x_1 + h/2, y_1 + k_1/2]$

$= (0.1) f(0.15, 1.1849) = 0.1582$.

$K_3 = hf(0.5, y_1 + k_3/2)$

$= (0.1) f(0.15, 1.196)$

$= 0.16098$

$k_4 = (0.1) f(0.2, 1.2779)$

$= 0.1889.$

(i.e) $y(0.2) = 1.2774$

Start from (x_2, y_2) to get y_3

$K_1 = hf(x_2, y_2)$

$= (0.1) f(0.2, 1.2774)$

$= 0.1884.$

$k_2 = hf(x_2 + h/2, y_2 + k_1/2)$

$= (0.1) f(0.25, 1.3178)$

$= 0.2225$

$K_3 = hf(x_2 + h/2, y_2 + k_2/2)$

$= (0.1) f(0.25, 1.3887)$

$= 0.2274.$

$K_4 = hf(x + h, y_2 + k_3) = (0.1) f(0.3, 1.5048)$

$= 0.2716$

$y_3 = [1.2774] + \frac{1}{6}[0.1884 + 2(0.2225) + 2(0.2274)$
$+ 0.2716]$

$= 1.5041.$

Now we use Adam's predictor formula.

$y_{4,p} = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0']$ —— (1)

$y_0' = x_0y_0 + y_0^2 = 1$

$y_1' = x_1y_1 + y_1^2 = 1.3592$

$y_2' = x_2y_2 + y_2^2 = 1.8872$

$y_3' = x_3y_3 + y_3^2 = 2.7135$

using (1),

$y_{4,p} = 1.5041 + \frac{0.1}{24}[55(2.7135) - 59(1.8872) + 37(1.3592) - 9(1)]$

$= 1.8341$

$y'_{4,p} = x_4y_4 + y_4^2$

$= (0.4)[1.8341) + (1.8341)^2]$

$= 4.0976.$

Adams corrector formula is

$y_{4,c} = y_3 + h/_{24}[9y_4' + 19y_3' - 5y_2' + y_1']$

$= 1.5041 + \frac{0.1}{24}[9(4.0976) + 19(2.7135) - 5(1.8872) + 1.3592]$

$= 1.8389$ //

$y(0.4) = 1.8389$ //