UNIT - V

Boundary value problems in ordinary and partial differential equations

Finite difference of 2nd order O.D.E

Consider a second order O.D.E given by $f(x, y, y', y'')=0$ with boundary conditions specified at $x=a$ & $x=b$.

Divide the interval $[a,b]$ into 'n' sub-intervals, each of length $h=\frac{b-a}{n}$. Let $x_i = x_0 + ih$, $i=0,1,2 \dots n$, where $x_0=a$ & $x_n=b$.

We use the notations $y_i = y(x_i)$, $y'(x_i)=y_i'$ & $y''(x_i)=y_i''$.

The finite difference approximations to the derivatives are given by

$$y_i' = \frac{y_{i+1}-y_{i-1}}{2h} \quad \text{and}$$

$$y_i'' = \frac{y_{i+1}-2y_i+y_{i-1}}{h^2}.$$

Substituting the above approximations for $y'(x)$ & $y''(x)$ and setting $i=1,2,3 \dots n-1$ we get a system of eqns. for the $(n-1)$ unknowns. Solving the system, the values $y_1, y_2 \dots y_{n-1}$ are known. From the boundary conditions, $y_0=y(a)$ & $y_n=y(b)$ are known.

1. Solve $y'' - y = x$, $x \in (0,1)$ given $y(0) = y(1) = 0$ using finite differences dividing the interval into 4 equal parts.

Soln:

Since $h = \frac{b-a}{n} = \frac{1}{4}$.

The nodal pts. are $x = 0, \frac{1}{4}, \frac{2}{4}, 3/4, 1$.

Given $y''(x) - y(x) = x$ ——— (1)

Using the central difference approximation we have

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$$\therefore (1) \Rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$$\therefore (1) \Rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - y_i = x_i$$

$y_{i+1} - (2+h^2)y_i + y_{i-1} = h^2 x_i$ where $h = 1/4$.

$\therefore 16\, y_{i+1} - 33 y_i + 16 y_{i-1} = x_i$ Where $i = 1,2,3$

Setting $i = 1,2,3$ & using $y_0 = y(0) = 0$, $y_4 = y(1) = 0$.

$16 y_2 - 33 y_1 = \frac{1}{4}$ Since $x_1 = 1/4$ ——— (2)

$16 y_3 - 33 y_2 + 16 y_1 = 1/2$ & $x_2 = 1/2$ ——— (3)

$-33 y_3 + 16 y_2 = 3/4$ since $x_3 = 3/4$ ——— (4)

(2) - (4) gives $y_1 - y_3 = \frac{1}{66}$ ——— (5)

Eliminating y_2 from (2) & (3), we get

$$256 y_3 - 833 y_1 = \frac{65}{4} \quad \text{——— (6)}$$

Eliminating y_3 from (5) & (6) we get

$$y_1 = -0.03488$$

$$y_3 = (833 y_1 + 65/4) / 256$$

(ie) $y_3 = -0.05002$. $\Rightarrow y_2 = \frac{334 y_1 + 1/4}{16} = -0.05632$ //

Tabulating, we have,

x	0	0.25	0.5	0.75	1
y	0	-0.03488	-0.05632	-0.05002	0

2) Solve $y'' - xy = 0$, given $y(0) = -1$, $y(1) = 2$ by finite difference method taking $n = 2$.

Soln:

If $n=2$, then $h = \frac{b-a}{n} = \frac{1-0}{2} = 1/2$ Since range $(0,1)$

The nodal pts are $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$

Given $y'' - xy = 0$ ——— (1)

Using central difference approximation, we have

$$y'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$(1) \Rightarrow \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - x_i y_i = 0$$

(ie) $y_{i+1} - (2 + h^2 x_i) y_i + y_{i-1} = 0$

Where $i = 1$, $h = 1/2$, $x_i = 0.5$, $y_0 = -1$, $y_2 = 2$.

$$\therefore y_2 - (2 + \tfrac{1}{8}) y_1 + y_0 = 0$$

$$2 - \frac{17}{8} y_1 - 1 = 0 \Rightarrow y_1 = 8/17 = 0.4706$$

Tabulating, we get

x	0	0.5	1
y	-1	0.4706	2

(3) Using the finite difference method, find $y(0.25)$, $y(0.5)$ and $y(0.75)$ satisfying the differential equation $\frac{d^2y}{dx^2} + y = x$. Subject to the boundary conditions $y(0) = 0$, $y(1) = 2$.

Solu:

The given diff. equation can be written as

$$y''(x) + y(x) = x \quad \text{--- (1)}$$

Using the central difference approximation we have

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \quad \text{--- (2)}$$

$$(1) \Rightarrow \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + y_i = x_i$$

$$y_{i+1} - (2 - h^2) y_i + y_{i-1} = h^2 x_i$$

$$y_{i+1} - \left[2 - \frac{1}{16}\right] y_i + y_{i-1} = \frac{1}{16} x_i \quad \left[\because h = \frac{1}{4}\right]$$

$$y_{i+1} - \frac{31}{16} y_i + y_{i-1} = \frac{1}{16} x_i$$

$$y_{i+1} - \frac{31}{16} y_i + y_{i-1} = 0.0625 x_i$$

Given $y_0 = 0$, $x_0 = 0$, $y_4 = 2$, $x_4 = 1$

$x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$.

If $i = 1$ then (3) $\Rightarrow y_0 - 1.9375 y_1 + y_2 = 0.0625 x_1$

$$-1.9375 y_1 + y_2 = 0.0156 \quad \text{--- (4)}$$

If $i=2$, then (3) $\Rightarrow y_1 - 1.9375y_2 + y_3 = 0.0625x_2$

$$y_1 - 1.9375y_2 + y_3 = 0.0313 \quad \text{--- (5)}$$

If $i=3$, then (3) $\Rightarrow y_2 - 1.9375y_3 + y_4 = 0.0625x_3$

$$y_2 - 1.9375y_3 = -1.9531 \quad \text{--- (6)}$$

(4) $\times$ (1) $\Rightarrow -1.9375y_1 + y_2 = 0.0156$

(5) $\times 1.9375 \Rightarrow 1.9375y_1 - 3.7539y_2 + 1.9375y_3 = 0.0606$

$$-2.7539y_2 + 1.9375y_3 = 0.0762 \quad \text{--- (7)}$$

(6) $\times 1 \Rightarrow y_2 - 1.9375y_3 = -1.9531$

(7) $\times 1 \Rightarrow -2.7539y_2 + 1.9375y_3 = 0.0762$

$$-1.7539y_2 = -1.8769$$

$$y_2 = \frac{1.8769}{1.7539} = 1.0701$$

(6) $\Rightarrow y_2 - 1.9375y_3 = -1.9531$

$$-1.9375y_3 = -1.9531 - y_2$$

$$y_3 = \frac{3.0232}{1.9375} = 1.5604.$$

(4) $\Rightarrow -1.9375y_1 + 1.0701 = 0.0156$

$$-1.9375y_1 = 0.0156 - 1.0701$$

$$= -1.0545$$

$$y_1 = \frac{1.0545}{1.9375} = 0.5443$$

Tabulating the values, the solution is

x	0	0.25	0.5	0.75	1
y	0	0.5443	1.0701	1.5604	2.

Finite difference solution of one-D heat equation by implicit & Explicit Method,

Classification of pde of 2^{nd} order.

The most general linear p.de of 2nd order can be written as $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$.
where A, B, C, D, E, F are in general func. of x & y.
The above equ. of 2nd is said to

(i) elliptic if $B^2 - 4AC < 0$

(ii) parabolic if $B^2 - 4AC = 0$

(iii) hyperbolic if $B^2 - 4AC > 0$.

Note: The same diff. equ. may be elliptic in one region, parabolic in another & hyperbolic in some other region.

For example:

1) $$xU_{xx} + U_{yy} = 0$$

Here $A = x,\ B = 0,\ C = 1$

$B^2 - 4AC = -4x$

$xU_{xx} + U_{yy} = 0$ is elliptic if $x > 0$,
hyperbolic if $x < 0$
& parabolic if $x = 0$

2) $xf_{xx} + yf_{yy} = 0,\ x > 0,\ y > 0$, classify the pde.

$A = x,\ B = 0,\ C = y \Rightarrow B^2 - 4AC = -4xy = -ve$

$\therefore$ It is elliptic $\forall x > 0, y > 0$. $(x > 0, y > 0$ given$)$

Solution of one-D heat equation

Bender-Schmidt's Difference Method (Explicit method)

Consider the one-D heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$.

This is an example of parabolic eqn. where ($\alpha^2 = k/\rho c$)

Setting $\alpha^2 = \frac{1}{a}$, the equation becomes,

$$\frac{\partial^2 u}{\partial x^2} - a \frac{\partial u}{\partial t} = 0.$$

To solve this equation by the method of finite differences $u_{xx} = a u_t$ —— (1)

with boundary conditions $u(0,t) = T_0$, $u(l,t) = T_1$ —— (2)

and with initial condition

$$u(x,0) = f(x), \quad 0 < x < l \quad \text{—— (3)}$$

We select a spacing h for the variable x and a spacing k for the time variable t.

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$\& \quad u_t = \frac{u_{i,j+1} - u_{i,j}}{k}$$

Substituting these in (1), we have

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = \frac{a}{k}(u_{i,j+1} - u_{i,j}).$$

$$\therefore u_{i,j+1} - u_{i,j} = \frac{k}{ah^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

$$= \lambda (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

where $\lambda = \frac{k}{ah^2}$

(ie) $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$ —— (4)

Writing the boundary conditions as

$u_{0,j} = T_0$ —— (5)

$u_{n,j} = T_1$ —— (6)

where $nh = l$

(i) initial condition as $u_{i,0} = f(ih)$, $i = 1, 2 \ldots$ —— (7)

U is known as $t = 0$.

Equ. (5) facilitates to get the value of u at $x = ih$ & time $t_j + k$.

Equ. (5) is called Explicit formula. It is valid if $0 < \lambda < 1/2$.

If we take, $\lambda = 1/2$, the coeff. of $u_{i,j}$ vanishes. Hence equ. (5) becomes,

$u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}]$ —— (8)

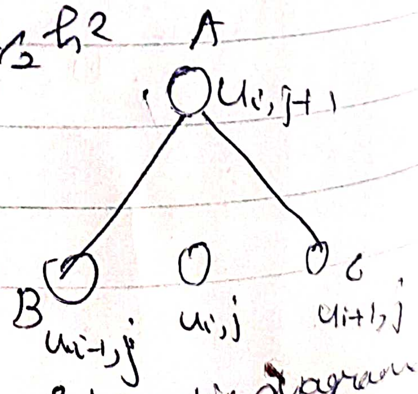
where $\lambda = 1/2 = k/ah^2$, (ie) $k = \frac{a}{2}h^2$

(ie) the value of u at $x = x_i$ at $t = t_{j+1}$ is equal to the average of the values of u the surrounding pts. x_{i-1} & x_{i+1} at previous time t_j.

Equ. (8) is called Bender-Schmidt recurrence equ. This is valid only if $k = a/2\, h^2$

Value of u at A

$= \frac{1}{2}$[value of u at B + value of u at C]



Schematic diagram

I. State Schmidt's explicit formula for solving heat flow equation.

$$u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda) u_{i,j} + \lambda u_{i-1,j}$$

If $\lambda = \frac{1}{2}$ $\Rightarrow$ $u_{i,j+1} = \frac{1}{2}[u_{i+1,j} + u_{i-1,j}]$.

1. Solve $u_{xx} = 32u_t$, taking $h = 0.25$ for $t > 0$, $0 < x < 1$ & $u(x,0) = 0$, $u(0,t) = 0$, $u(1,t) = t$.

Solu:

The range for x is $(0,1)$; Given $h = 0.25$

k is not given. For applying Bender-Schmit method

$$k = \frac{a h^2}{2} = \frac{32}{2}\left(\frac{1}{4}\right)^2 = 1$$

step size of time t is 1.

The formula is $u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}]$ — (1)

Using (1), the values of up to $t = 5$ sect are tabulated below

direction of x

$j \backslash i$	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	0.5	2
3	0	0	0.25	1	3
4	0	0.125	0.5	0.625	4
5	0	0.25	0.875	2.25	5

t-direction

values of u.

2) Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 1$, $t > 0$ with $u(x,0) = x(1-x)$, $0 < x < 1$ and $u(0,t) = u(1,t) = 0$, $\forall\, t > 0$ using explicit method with $\Delta x = 0.2$ for 5 time steps.

Solu:

Given $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

Here $\alpha^2 = 1$, $\Delta x = 0.2$ (i.e.) $h = 0.2$

k is not given.

Let us choose k such that

$\lambda = \frac{k\alpha^2}{h^2} = \frac{1}{2}$ [Explicit formula valid if $0 < \lambda \le \frac{1}{2}$]

$\frac{k}{(0.2)^2} = \frac{1}{2}$ [So choose $\lambda = \frac{1}{2}$]

$k = \frac{0.04}{2} = 0.02$

$$u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$$

$$= \frac{1}{2}[u_{i+1,j} + u_{i-1,j}] \quad \text{--- (1)}$$

Given $u(x,0) = x(1-x)$

$u(0.2, 0) = (0.2)(1-0.2) = 0.16$

$u(0.4, 0) = 0.24$

$u(0.6, 0) = 0.24$

$u(0.8, 0) = 0.16$

Using (1) the values of u are tabulated below.

$j \backslash i$	0	0.2	0.4	0.6	0.8	1.0
0	0	0.16	0.24	0.24	0.16	0
0.02	0	0.12	0.2	0.2	0.12	0
0.04	0	0.1	0.16	0.16	0.1	0
0.06	0	0.08	0.13	0.13	0.08	0
0.08	0	0.065	0.105	0.105	0.065	0
0.10	0	0.0525	0.085	0.085	0.0525	0 Given $u(l,t)=0$

Given $u(0,t)=0$.

3) Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ given $u(0,t)=0$, $u(4,t)=0$
$u(x,0) = x(4-x)$ assuming $h=k=1$. Find the values of u upto $t=5$.

Soln:

If we want to use Bender-Schmidt formula, we should have $k = \frac{a}{2}h^2$.

Here $k=h=1$, $a=1$. These values do not satisfy the condition. Hence we cannot apply Bender-Schmidt formula. Hence we apply Explicit formula.

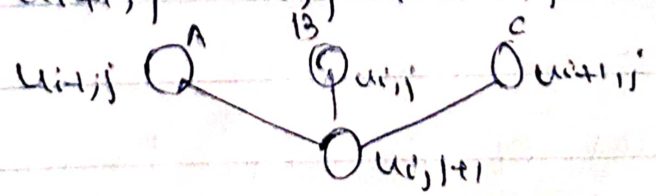
(ie) $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$ ——— (1)

Now $\lambda = \frac{k}{ah^2} = \frac{1}{1\times 1} = 1$

Hence eqn. (1) reduces to

$u_{i,j+1} = u_{i+1,j} - u_{i,j} + u_{i-1,j}$ ——— (2)

That is



Value of u at D = Value of u at A + Value of u at C − value of u at B.

Using (2), the values of u are tabulated below

x direction

j\i	0	1	2	3	4
0	0	3	4	3	0
1	0	1	2	1	0
2	0	1	0	1	0
3	0	−1	2	−1	0
4	0	3	−4	3	0
5	0	−7	10	−7	0

t-direction

4) Solve $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0$ given $u(0,t)=0$, $u(4,t)=0$, $u(x,0) = x(4-x)$. Assume $h=1$. Find the value of u up to $t=5$.

Soln: Given $u_{xx} = a u_t$ $\therefore a = 2$

To use Bender-Schmidt's eqn. $k = \frac{a}{2}h^2 = 1$

step-size in time $= k = 1$.

Given Range for x; (0,4) for t (0,5)

$u(x,0) = x(4-x) \Rightarrow u(0,0)=0$, $u(1,0)=3$

$u(2,0)=4$, $u(3,0)=3$,

$u(4,0)=0$.

Also given $u(0,t)=0$, $u(4,t)=0$ $\forall t$

The values of $u_{i,j}$ are tabulated below.

x-direction

t-direction

j\i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

Crank-Nicholson - Difference Method [Implicit Method]

Aim: to solve the parabolic equation.

Consider the one-D heat equation

$u_{xx} = a u_t$ with boundary conditions.

$u(0,t) = T_0$, $u(l,t) = T_l$ and the initial condition

$u(x,0) = f(x)$, $0 < x < l$.

The equation to be solved is $u_{xx} = a u_t$ —— (1)

At $u_{i,j}$

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

at $u_{i,j+1}$

$$u_{xx} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}$$

Taking the average of these two values

$$u_{xx} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j+1} + u_{i-1,j}}{2h^2}$$

Using $U_t = \dfrac{u_{i,j+1} - u_{i,j}}{k}$ equ. (1) reduces to

$$\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{2h^2} = a\,\frac{u_{i,j+1} - u_{i,j}}{k}.$$

Setting $\dfrac{k}{ah^2} = \lambda$, the above equ. reduces to

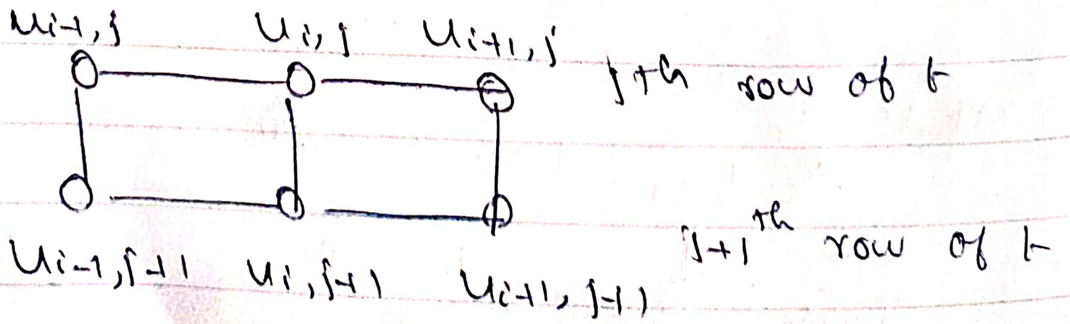
$$\tfrac{1}{2}\lambda u_{i+1,j+1} + \tfrac{1}{2}\lambda u_{i-1,j+1} - (\lambda+1) u_{i,j+1}$$
$$= -\tfrac{1}{2}\lambda u_{i+1,j} - \tfrac{1}{2}\lambda u_{i-1,j} + (\lambda-1) u_{i,j} \quad \text{—— } \cancel{(I)}$$

This can be written as

$$\lambda(u_{i+1,j+1} + u_{i-1,j+1}) - 2(\lambda+1) u_{i,j+1}$$
$$= 2(\lambda-1) u_{i,j} - \lambda(u_{i+1,j} + u_{i-1,j}) \quad \text{—— (I)}$$

Equ. I is called Crank-Nicholson difference scheme (or) Method.

Note: 1: The six points in the above formula are shown below

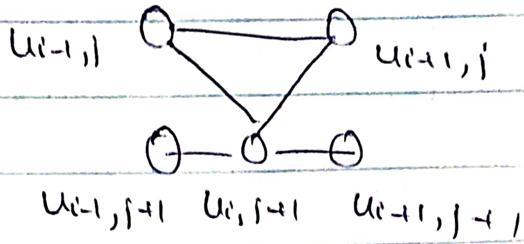


Note: 2

If $\lambda = 1$ (ie) $k = ah^2$, then the Crank-Nicholson reduces to

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right]$$

Schematic diagram.



The value of u at A = average of the values at B, C, D,

1) Solve by Crank - Nicholson method the equation $u_{xx} = u_t$. Subject to $u(x,0) = 0$, $u(0,t) = 0$ & $u(1,t) = t$, for two time steps.

Soln.

x ranges from 0 to 1, take $h = 1/4$;

here $a = 1$

$\therefore k = ah^2$ to use simple form

$K = 1\left(\frac{1}{4}\right)^2 = \frac{1}{16}$

We use $u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j}\right]$ —(

Given B-Cs, $u(0,t) = 0$ & $u(1,t) = t$

& Initial condition is $u(x,0) = 0$.

x - direction

x - direction

		0	0.25	0.5	0.75	1
	0	0	0	0	0	0
y - direction	$\frac{1}{16}$	0	u_1	u_2	u_3	$\frac{1}{16}$
	$\frac{2}{16}$	0	u_4	u_5	u_6	$\frac{2}{16}$

Let the unknown be represented by u_1, u_2, u_3 ...

(1) ⇒ $u_1 = \frac{1}{4}(0+0+0+u_2)$ (i.e.) $u_1 = \frac{1}{4}u_2$ — (2)

$u_2 = \frac{1}{4}(0+0+u_1+u_3)$ $u_2 = \frac{1}{4}(u_1+u_3)$ — (3)

$u_3 = \frac{1}{4}(0+0+u_2+\frac{1}{16})$ $u_3 = \frac{1}{4}(u_2+\frac{1}{16})$ — (4)

Sub u_1, u_3 values in equ. (3)

$$u_2 = \frac{1}{4}\left[\frac{1}{4}u_2 + \frac{1}{4}\left(u_2+\frac{1}{16}\right)\right]$$

$$= \frac{1}{16}\left[u_2+u_2+\frac{1}{16}\right]$$

$$= \frac{1}{16}\left[2u_2+\frac{1}{16}\right] = \cancel{u_2}$$

$$= \frac{u_2}{8} + \frac{1}{16}\cdot\frac{1}{16}$$

$$u_2 - \frac{u_2}{8} = \frac{1}{16}\cdot\frac{1}{16}$$

$$\frac{7u_2}{8} = \frac{1}{16}\cdot\frac{1}{16}$$

$$u_2 = \frac{1}{16}\cdot\frac{1}{16}\times\frac{8}{7} = 0.0045$$

Sub $u_2 = 0.0045$ in equ. (2)

$u_1 = 0.0011$

Sub $u_2 = 0.0045$ in eqn (4)

$$u_3 = \frac{1}{4}\left(0.0045 + \frac{1}{16}\right)$$

$$= 0.0168$$

From table

$u_4 = \frac{1}{4}[0+0+u_2+u_5]$ (i.e.) $u_4 = \frac{1}{4}(0.0045 + u_5)$ —— (5)

$u_5 = \frac{1}{4}(u_1+u_3+u_4+u_6)$ $\quad u_5 = \frac{1}{4}(0.0179 + u_4 + u_6)$ —— (6)

$u_6 = \frac{1}{4}\left(u_2 + \frac{1}{16} + u_5 + \frac{2}{16}\right)$ $\quad u_6 = \frac{1}{4}(0.192 + u_5)$ —— (7)

Sub. u_4, u_6 values in eqn (6)

$$u_5 = \frac{1}{4}\left(0.0179 + \frac{1}{4}(0.0045+u_5) + \frac{1}{4}(0.192+u_5)\right]$$

$$= \frac{1}{4} \times \frac{1}{4}\left[4 \times 0.0179 + 0.0045 + 0.192 + 2u_5\right]$$

$$= \frac{1}{16}\left[0.2681 + 2u_5\right]$$

$$= 0.01675625 + \frac{1}{8}u_5$$

$$u_5 - \frac{1}{8}u_5 = 0.01675625$$

$$u_5 = 0.01675625 \times \frac{8}{7} = 0.01915$$

(5) $\Rightarrow u_4 = \frac{1}{4}(0.0045 + 0.01915)$

$$= 0.00591$$

(7) $\Rightarrow u_6 = \frac{1}{4}(0.192 + 0.01913)$

$$= 0.052787$$

$u_4 = 0.005899$

$u_5 = 0.01913$

$u_6 = 0.05277$.

1) Write down the Crank-Nicolson formula to solve $u_t = u_{xx}$. (or) Write down the implicit form to solve one-D heat flow equation.

Soln.

$$\frac{1}{2}\lambda u_{i+1,j+1} + \frac{1}{2}\lambda u_{i-1,j+1} - (\lambda+1)u_{i,j+1}$$
$$= -\frac{1}{2}\lambda u_{i+1,j} - \frac{1}{2}\lambda u_{i-1,j} + (\lambda-1)u_{i,j}$$

(or) $$\lambda(u_{i+1,j+1} + u_{i-1,j+1}) - 2(\lambda+1)u_{i,j+1}$$
$$= 2(\lambda-1)u_{i,j} - \lambda(u_{i+1,j} + u_{i-1,j})$$

2) What type of equations can be solved by Crank-Nickolson's difference formula

Crank-Nickolson's difference formula is used to solve parabolic equations of the form $u_{xx} = a u_t$.

2) Using Crank-Nicholson's scheme, solve $u_{xx} = 16u_t$, $0<x<1$, $t>0$ given $u(x,0)=0$, $u(0,t)=0$, $u(1,t)=100t$. Compute u for one step in t-direction taking $h=\frac{1}{4}$.

Soln:

Here $a=16$, $h=\frac{1}{4}$

$\therefore k = ah^2$ to use simple form

$$k = 16\left(\frac{1}{16}\right) = 1$$

$\therefore$ We use $u_{i,j+1} = \frac{1}{4}[u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j}]$

Given B-Cs are $u(0,t)=0$, $u(1,t)=100t$

Given initial condition is $u(x,0)=0$

j\i	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	u_1	u_2	u_3	100

Using (1)

$u_1 = \frac{1}{4}(0+0+0+u_2)$ (or) $u_1 = \frac{1}{4}u_2$ — (2)

$u_2 = \frac{1}{4}(0+0+u_1+u_3)$ $u_2 = \frac{1}{4}(u_1+u_3)$ — (3)

$u_3 = \frac{1}{4}(0+0+u_2+100)$ $u_3 = \frac{1}{4}(u_2+100)$ — (4)

Sub. u_1, u_3 values in (3),

$$u_2 = \frac{1}{4}\left[\frac{1}{4}u_2 + \frac{1}{4}(u_2+100)\right]$$

$$= \frac{1}{16}(u_2+u_2+100)$$

$$= \frac{1}{8}u_2 + \frac{25}{4}$$

$$u_2 - \frac{1}{8}u_2 = \frac{25}{4}$$

$$7/8\, u_2 = 25/4$$

$$u_2 = \frac{25}{4} \times 8/7 = \frac{50}{7} = 7.1429$$

Sub. $u_2 = 7.1429$ in (2), $u_1 = 1.7857$.

Sub $u_2 = 7.1429$ in (4), $u_3 = 26.7857$.

$\therefore u_1 = 1.7857$, $u_2 = 7.1429$, $u_3 = 26.7857$.

5) Using Crank-Nicholson method, solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, subject to $u(x,0)=0$, $u(0,t)=0$ & $u(1,t)=t$ (i) taking $h=0.5$ & $k=\frac{1}{8}$ (ii) $h=\frac{1}{4}$ & $k=1/8$

Soln:

(i) Here $a=1$; $h=0.5$ & $k=\frac{1}{8}$

To use simplified form, $k=ah^2$, but here $k \neq ah^2$

$\therefore$ we use

$$\lambda(u_{i+1,j+1}+u_{i-1,j+1}) - 2(\lambda+1)u_{i,j+1}$$
$$= 2(\lambda-1)u_{i,j} - \lambda(u_{i+1,j}+u_{i-1,j}) \quad \text{--- (1)}$$

where $\lambda = \frac{k}{ah^2}$.

$\therefore \lambda = \frac{1}{8\times 1 \times (0.5)^2} \Rightarrow \lambda = \frac{1}{2}$ sub in (1)

$$\frac{1}{2}(u_{i+1,j+1}+u_{i-1,j+1}) - 2(\tfrac{1}{2}+1)u_{i,j+1}$$
$$= 2(\tfrac{1}{2}-1)u_{i,j} - \frac{1}{2}(u_{i+1,j}+u_{i-1,j})$$

$$\Rightarrow u_{i+1,j+1}+u_{i-1,j+1} - 6u_{i,j+1} = -2u_{i,j} - (u_{i-1,j}+u_{i+1,j}) \quad \text{--- (2)}$$

x direction

$j \backslash i$ (t-direction)	0	0.5	1
0	0	0	0
$\frac{1}{8}$	0	u_1	$-\frac{1}{8}$

Now using (2)

$$\frac{1}{8} + 0 - 6u_1 = -2(0) - (0+0)$$

$$\therefore u_1 = \frac{1}{48} = 0.0208333$$

ii) Given $k = \frac{1}{8}$, $h = \frac{1}{4}$.

$$\Rightarrow \lambda = \frac{k}{ah^2} = \frac{\frac{1}{8}}{1\times\frac{1}{16}} = 2.$$

Since $\lambda = 2$, we cannot use simplified form

$\therefore$ using in general Crank-Nicholson formula.

cre) $\lambda(u_{i+1,j+1}+u_{i-1,j+1})-2(\lambda+1)u_{i,j+1}$

$$=2(\lambda-1)u_{i,j}-\lambda(u_{i+1,j}+u_{i-1,j}) \text{ ——— } (\mathrm{I})$$

put $\lambda=2$ in (I) we get

$2(u_{i+1,j+1}+u_{i-1,j+1})-2(2+1)u_{i,j+1}$

$$=2(2-1)u_{i,j}-2(u_{i+1,j}+u_{i-1,j})$$

$\div 2$, $u_{i+1,j+1}+u_{i-1,j+1}-3u_{i,j+1}=u_{i,j}-u_{i+1,j}-u_{i+1,j}$ — II

x direction

t-direction	0	0.25	0.5	0.75	1
0	0	0	0	0	0
$\frac{1}{8}$	0	u_1	u_2	u_3	$\frac{1}{8}$

$0+u_2-3u_1=0$ $\quad\therefore u_2=3u_1$ ——— (4)

$u_1+u_3-3u_2=0$ $\quad 3u_2=u_1+u_3$ ——— (5)

Also $u_2+\frac{1}{8}-3u_3=0$. $\quad 3u_3=u_2+\frac{1}{8}$ ——— (6)

Adding (2) & (4)

$$3(u_1+u_3)=2u_2+\frac{1}{8} \text{ ——— } (5)$$

From (8) & (5), we get

$$9u_2=2u_2+\frac{1}{8}.$$

$\therefore u_2=\frac{1}{56}=0.01786$

$u_1=\frac{1}{168}=0.00595$

$u_3=\frac{8}{168}=0.04762$ //

One dimensional wave equation

[Hyperbolic type]

Introduction

The one dimensional wave equation is of hyperbolic type.

Aim: To solve the hyperbolic equation.

Consider the one-D wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

(ie) $a^2 u_{xx} - u_{tt} = 0$ ——— (1)

Subject to the initial conditions $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = 0$ with the boundary conditions $u(0,t) = 0$, $u(l,t) = 0$.

Assuming $\Delta x = h$, $\Delta t = k$, we have

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$u_{tt} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

Substituting these values in (1) we get

$$\frac{a^2}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - \frac{1}{k^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = 0.$$

(ie) $\lambda^2 a^2 (u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - u_{i,j+1} + 2u_{i,j} - u_{i,j-1} = 0$

where $\lambda = k/h$.

$$\lambda^2 a^2 u_{i+1,j} - 2\lambda^2 a^2 u_{i,j} + \lambda^2 a^2 u_{i-1,j} - u_{i,j+1} + 2u_{i,j} - u_{i,j-1} = 0$$

$$u_{i,j+1} = 2(1-\lambda^2 a^2) u_{i,j} + \lambda^2 a^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \quad \text{——— (2)}$$

Equ. (2) is called an explicit formula to solve the wave equation.

To get a simpler form, we choose λ such that

$1-\lambda^2 a^2 = 0 \Rightarrow \lambda^2 = \frac{1}{a^2}$

$\therefore \frac{k^2}{h^2} = \frac{1}{a^2} \Rightarrow k = h/a.$

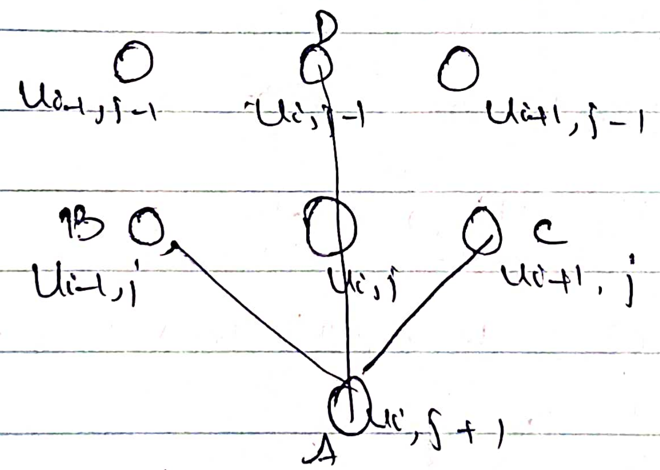
Hence if $k = h/a$ (or) $\lambda^2 = 1/a^2$, the explicit formula (2) takes the form.

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \quad \text{——— (3)}$$

Equ. (3) gives a simpler form under the condition $k = h/a$.

Note:

Schematic diagram



The value of u at A = value of u at (B+C−D)

1) Solve numerically $4u_{xx} = u_{tt}$ with the boundary condition, $u(0,t)=0$, $u(4,t)=0$ and the initial conditions $u_t(x,0)=0$ & $u(x,0)=x(4-x)$, taking $h=1$ and up to $t=5$ seconds,

Soln:

Here $a^2 = 4$, $h = 1$.

To use simpler form, take $k = \frac{h}{a} = \frac{1}{2}$. The simplest form of explicit scheme is

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \quad \text{---} (1)$$

Given Boundary conditions are

$$u(0,t) = 0, \ \forall t, \qquad u(4,t) = 0, \ \forall t$$

Given initial conditions are,

$$u(x,0) = x(4-x) \text{ --- } (2), \quad u_t(x,0) = 0 \text{ --- } (3)$$

(2) $\Rightarrow u(0,0) = 0,\ u(1,0) = 3,\ u(2,0) = 4,\ u(3,0) = 3,\ u(4,0) = 0$

From (3) $u_t(x,0) = 0 \Rightarrow u_{i,1} = \frac{u_{i+1,0} + u_{i-1,0}}{2}$

put $i = 1, 2, 3$, we have

$$u_{1,1} = \frac{u_{2,0} + u_{1,0}}{2} = \frac{4+0}{2} = 2$$

$$u_{2,1} = \frac{u_{3,0} + u_{1,0}}{2} = \frac{3+3}{2} = 3$$

$$u_{3,1} = \frac{u_{4,0} + u_{2,0}}{2} = \frac{0+4}{2} = 2$$

$u_{4,1} = 0$. using (1), the values of u are tabulated below

$j \backslash i$	0	1	2	3	4
0	0	3	4	3	

$j \backslash i$	0	1	2	3	4
0	0	3	4	3	0
0.5	0	2	3	2	0
1	0	0	0	0	0
1.5	0	-2	-3	-2	0
2	0	-3	-4	-3	0
2.5	0	-2	-3	-2	0
3	0	0	0	0	0
3.5	0	2	3	2	0
4	0	3	4	3	0

$u(x,0) = x(4-x)$

Note: $u_t(x,0) = 0 \Rightarrow u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$

2) Solve $25u_{xx} - u_{tt} = 0$, given $u(0,t) = u(5,t) = 0$, $u_t(x,0) = 0$ & $u(x,0) = 2x$ for $0 \le x \le 2.5$
$= 10 - 2x$ for $2.5 \le x \le 5$
fore one half period of vibration.

Soln:

Here $a^2 = 25$ $\therefore a = 5$;

Period of vibration $= \frac{2l}{a} = \frac{2 \times 5}{5} = 2$ seconds.

half period $= 1$ second.

Therefore we want values upto $t = 1$ sec.

To use simple form, take $k = h/a = 1/5$.

taking $h = 1$, step in t-direction $= 1/5$.

The simplest form of explicit scheme is

$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

Given B-Cs are $u(0,t)=0$ (or) $u_{0,j}=0$
$u(5,t)=0$ (or) $u_{5,j}=0$ $\forall j$

Given I-Cs are

$$u(x,0) = \begin{cases} 2x & \text{for } 0 \le x \le 2.5 \\ 10-2x & \text{for } 2.5 \le x \le 5 \end{cases} \quad \text{(2)}$$

& $u_t(x,0)=0 \Rightarrow u_{i,1} = \dfrac{u_{i+1,0}+u_{i-1,0}}{2}$ (3)

(2) $\Rightarrow u(0,0)=0$, $u(1,0)=2$, $u(2,0)=4$
$u(3,0)=4$, $u(4,0)=2$, $u(5,0)=0$.

(3) $\Rightarrow$ $u_{1,1} = \dfrac{u_{2,0}+u_{0,0}}{2} = 2.$

$u_{2,1} = \dfrac{u_{3,0}+u_{1,0}}{2} = \dfrac{4+2}{2} = 3$

$u_{3,1} = \dfrac{u_{4,0}+u_{2,0}}{2} = 3$

$u_{4,1} = \dfrac{u_{5,0}+u_{3,0}}{2} = 2$

using (1), the values of u are tabulated below

j\i	0	1	2	3	4	5
0	0	2	4	4	2	0
1/5	0	2	3	3	2	0
2/5	0	1	1	1	1	0
3/5	0	-1	-1	-1	-1	0
4/5	0	-2	-3	-3	-2	0
1	0	-2	-4	-4	-2	0

Two dimensional Laplace equation

Elliptic equation : ($B^2-4AC<0$).

The Laplace equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and the poisson equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ are examples of elliptic partial differential equations.

Solution of Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——— (1)}$$

in (1) Replacing the derivatives by finite difference approximations, we get

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = 0 .$$

Taking $k=h$, (square mesh) in the above equation, we get

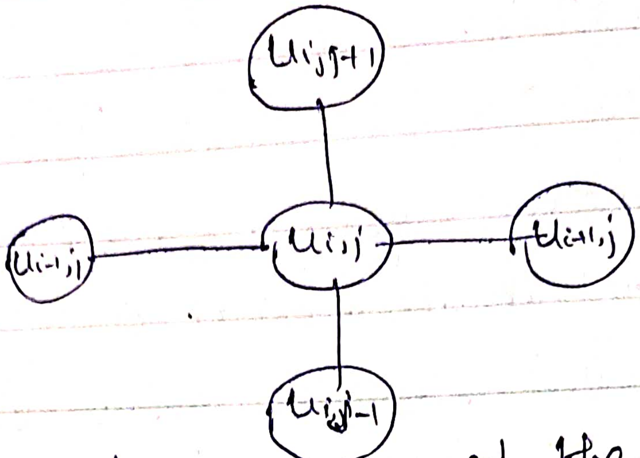
$$4u_{i,j} = u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}$$

$$\therefore u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right] \quad \text{——— (2)}$$

(ie) the value of u at any interior pt. is the arithmetic mean of the values of u ~~vertically~~ at the four lattice pts. (Two of them are ~~four~~ vertically just above & below and the other two in the horizontal line just after & before this point).

equ. (2) is called standard five pt. formula

Schematic diagram



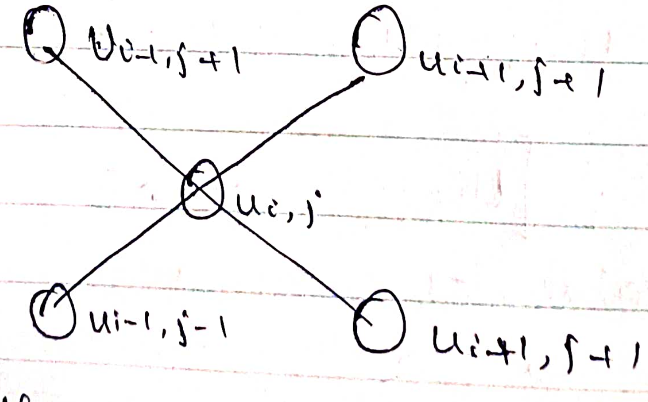
central value = average of the other four values

Diagonal five-pt. formula:

Instead of the formula (2), we can also use the formula.

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right] \quad \text{— (5)}$$

which is called the diagonal five-pt. formula. Since this formula involves the values on the diagonals through $u_{i,j}$



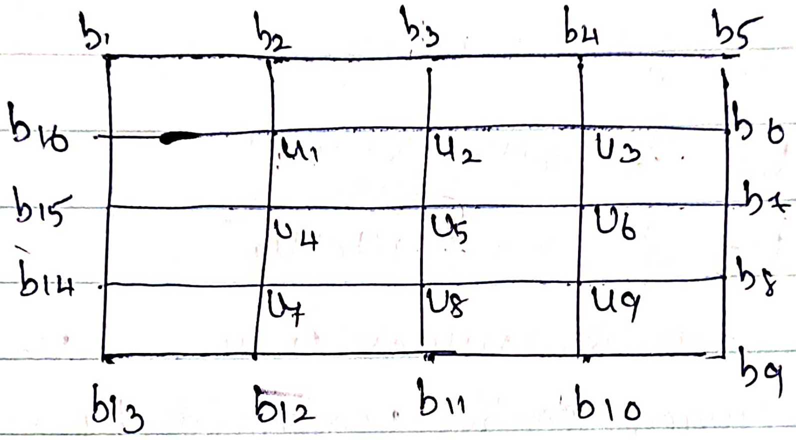
Note:

Since the error in the diagonal five pt. formula is four times the error in the std five pt. formula, we always prefer the std five pt. formula to the diagonal formula.

(By Liebmann's iteration process)

Consider the Laplace's equation $u_{xx}+u_{yy}=0$ in bounded square region R with a boundary C when the boundary values of u are given on the boundary.

Let us divide the square region into a network of sub-squares of side h.



The boundary values of u at the grid pts. are given and noted by $b_1, b_2 \dots b_{16}$. The values of u at the interior lattice (or) grid pts. are assumed to be $u_1, u_2 \dots u_9$.

To start the iteration process, initially we find rough values at interior pts. and then we improve them by iterative process mostly using standard five pt. formula.

Find u_5 first $u_5 = \frac{1}{4}(b_3+b_7+b_{11}+b_{15})$ (SFPF)

We compute u_1, u_3, u_7, u_9 by using diagonal five pt. formula (DFPF).

(ie) $u_1 = \frac{1}{4}(b_3 + b_{15} + b_1 + u_5)$

$u_3 = \frac{1}{4}(b_5 + u_5 + b_3 + b_7)$

$u_7 = \frac{1}{4}(u_5 + b_{13} + b_{11} + b_{15})$

$u_9 = \frac{1}{4}(b_7 + b_{11} + b_9 + u_5)$

The remaining 4 values u_2, u_4, u_6, u_8 can be got by using SFPF

$u_2 = \frac{1}{4}(b_3 + u_5 + u_1 + u_3)$

$u_4 = \frac{1}{4}(u_1 + u_7 + u_5 + b_{15})$

$u_6 = \frac{1}{4}(u_3 + u_9 + u_5 + b_7)$

$u_8 = \frac{1}{4}(u_5 + b_{11} + u_7 + u_9)$

Once all the values $u_1, u_2 \ldots u_9$ are computed their accuracy can be improved by iteration method. The iteration formula is given by

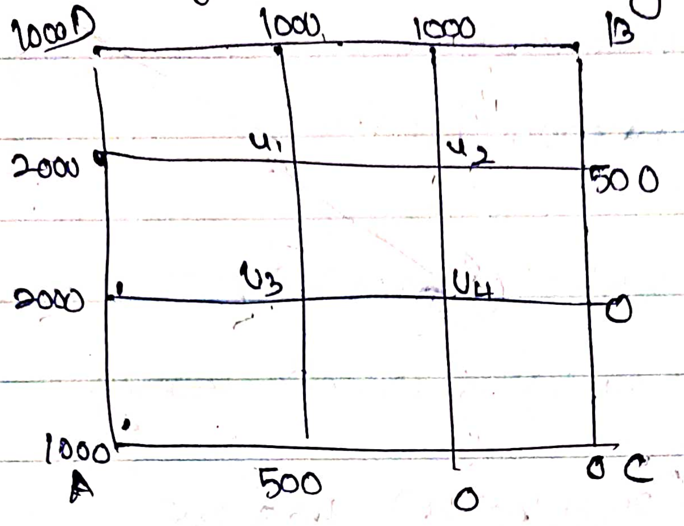
$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}\right]$$

where the superscript of u denotes the iteration number.

Equ. (I) is called Liebmann's iteration process. The process is stopped once we get the values with desired accuracy.

SFPF

1. Evaluate the fun. $f(x,y)$ satisfying $\nabla^2 u = 0$ at the lattice pt. given the boundary values as follows:



Soln:

We can assume some value for u_4 (or any other u): we can take $u_4 = 0$ and proceed (or) we take a value of $u_4 = 400$ (Given this seeing the values of u on the vertical line through u_2, u_4).

Rough values

$u_1 = \frac{1}{4}(1000 + 2000 + 1000 + u_4)$ DFPF

$\Rightarrow u_1 = \frac{1}{4}(1000 + 2000 + 1000 + 400) = 1100$

$u_2 = \frac{1}{4}(u_1 + u_4 + 1500)$

$= \frac{1}{4}(1100 + 400 + 1500) = 750$ SFPF

$u_3 = \frac{1}{4}(u_1 + 500 + 2000 + u_4)$

$= \frac{1}{4}(1100 + 500 + 2000 + 400) = 1000$ SFPF

$u_4 = \frac{1}{4}(u_2 + u_3)$

$= \frac{1}{4}(750 + 1000) = 437.5$ SFPF.

First iteration

Here after we apply only SFPF

$$u_1^{(1)} = \frac{1}{4}(2000+1000+u_2+u_3)$$

$$= \frac{1}{4}(2000+1000+750+1000)$$

$$= 1187.5$$

$$u_2^{(1)} = \frac{1}{4}(1187.5+437.5+1500) = 781.25$$

$$u_3^{(1)} = \frac{1}{4}(1187.5+437.5+2500) = 1031.25$$

$$u_4^{(1)} = \frac{1}{4}(781.25+1031.25) = 453.125$$

2nd iteration:

$$u_1^{(2)} = \frac{1}{4}(781.25+1031.25+3000)$$

$$= 1203.125$$

$$u_2^{(2)} = \frac{1}{4}(1203.125+453.125+1500) = 789.1$$

$$u_3^{(2)} = \frac{1}{4}(1203.125+453.125+2500)$$

$$= 1039.1$$

$$u_4^{(2)} = \frac{1}{4}(789.1+1039.1) = 457.1$$

3rd iteration:

$$u_1^{(3)} = \frac{1}{4}(789.1+1039.1+3000) = 1207.1$$

$$u_2^{(3)} = \frac{1}{4}(1207.1+457.1+1500) = 791.$$

$$u_3^{(3)} = \frac{1}{4}(1207.1+457.1+2500) = 1041.1$$

$$u_4^{(3)} = \frac{1}{4}(791.1+1041.1) = 458.1$$

Fourth iteration:

$$U_1^{(4)} = \frac{1}{4}(791.1 + 1041.1 + 3000) = 1208.1$$

$$U_2^{(4)} = \frac{1}{4}(1208.1 + 458.1 + 1500) = 791.6$$

$$U_3^{(4)} = \frac{1}{4}(1208.1 + 458.1 + 2500) = 1041.6$$

$$U_4^{(4)} = \frac{1}{4}(791.6 + 1041.6) = 458.3$$

Fifth iteration:

$$U_1^{(5)} = \frac{1}{4}(791.6 + 1041.6 + 3000) = 1208.3$$

$$U_2^{(5)} = \frac{1}{4}(1208.3 + 458.3 + 1500) = 791.7$$

$$U_3^{(5)} = \frac{1}{4}(1208.3 + 458.3 + 2500) = 1041.7$$

$$U_4^{(5)} = \frac{1}{4}(791.7 + 1041.7) = 458.4$$

We are getting result correct to one decimal place. Further the increase in the value is < 0.1

$\therefore u_1 = 1208.1,\ u_2 = 791.7,\ u_3 = 1041.7,\ u_4 = 458.7$

27) Solve $u_{xx} + u_{yy} = 0$ over the square mesh of side 4 units. Satisfying the following boundary conditions,

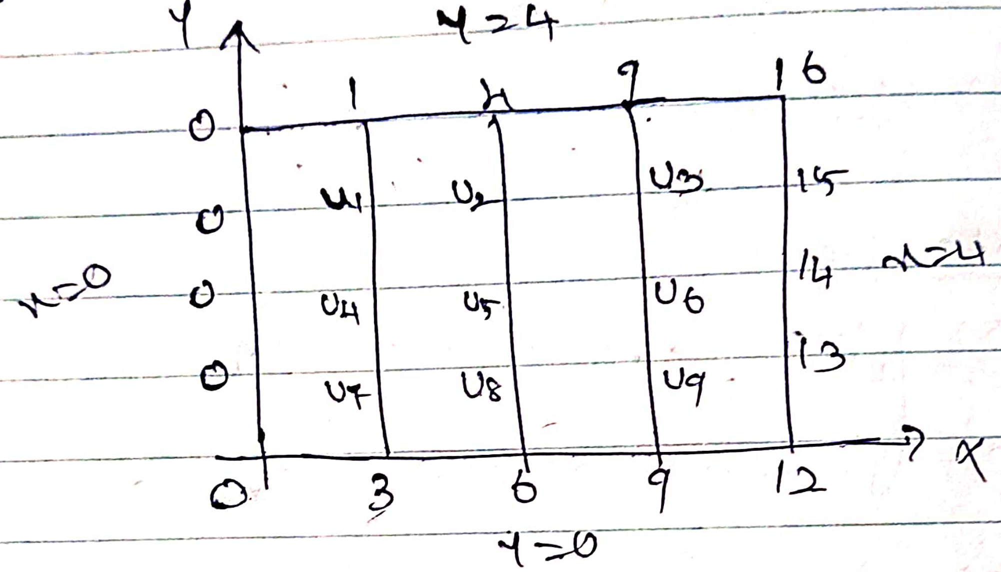
(i) $u(0,y) = 0$ for $0 \le y \le 4$.

(ii) $u(4,y) = 12+y$, for $0 \le y \le 4$

(iii) $u(x,0) = 3x$, for $0 \le x \le 4$

(iv) $u(x,4) = x^2$ for $0 \le x \le 4$

Soln. We divide the Square mesh into 16 Sub-Squares of side 1 unit and calculate the numerical values of u on the boundary using given analytical expression.



Let $u_1, u_2, \ldots u_9$ be the values of u at the interior nine grid pts.

Finding Rough values

$U_5 = \frac{1}{4}(4+6+0+14) = 6$ SFPF

$U_1 = \frac{1}{4}(0+6+4+0) = 2.5$ DFPF

$U_3 = \frac{1}{4}(16+6+14+4) = 10$ DFPF

$U_7 = \frac{1}{4}(0+6+0+6) = 3$ DFPF

$U_9 = \frac{1}{4}(6+14+6+12) = 9.5$ DFPF

We use SFPF to get the other values U

$U_2 = \frac{1}{4}(4+6+2.5+10) = 5.625$

~~$U_4 = \frac{1}{4}(0+6+2.5)+3 = 3.125$~~

$$U_4 = \frac{1}{4}(0+6+2.5+3) = 3.125$$

$$U_6 = \frac{1}{4}(6+14+10+9.5) = 9.875$$

$$U_5 = \frac{1}{4}(6+6+3+9.5) = 6.125$$

Now we proceed for iteration using always SFPF.

<u>Iteration Scheme.</u>

$$u_1 = \frac{1}{4}(1+u_2+u_4)$$

$$U_2 = \frac{1}{4}(4+u_1+u_3+u_5)$$

$$U_3 = \frac{1}{4}(24+u_2+u_6)$$

$$U_4 = \frac{1}{4}(u_1+U_5+U_7)$$

$$U_5 = \frac{1}{4}(U_2+U_4+U_6+U_8)$$

$$U_6 = \frac{1}{4}(14+U_3+U_5+u_9)$$

$$U_7 = \frac{1}{4}[3+U_4+U_8]$$

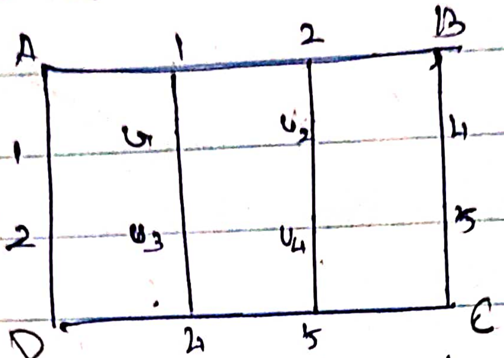
$$U_8 = \frac{1}{4}[6+U_5+U_7+u_9]$$

$$u_9 = \frac{1}{4}[22+U_8+U_6]$$

	u_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9
0	2.5	5.625	10	3.125	6	9.875	3	6.125	9.5
1	2.4375	5.6094	9.8711	2.8594	6.1172	9.8721	2.9961	6.153	9.5063
2	2.3672	5.5888	9.8652	2.8698	6.1209	9.8731	3.0057	6.1582	9.5078
3	2.3646	5.5874	9.8151	2.8728	6.1229	9.8739	3.0078	6.1596	9.5083

correct to two decimal places we have $u_1=2.37$, $u_2=5.59$, $u_3=9.87$, $u_4=2.88$, $u_5=6.13$, $v_6=9.88$, $u_7=3.01$, $U_8=6.16$, $u_9=9.51$

3. Solve $u_{xx}+u_{yy}=0$ for the following square mesh with boundary values as shown in the figure below.



Soln: Evidently the boundary values are symmetrical about the diagonal AC but not BD. Let the values at the interval grid pts. be u_1, u_2, u_3, u_4

By symmetry $u_2 = u_3$; $u_1 \neq u_4$, we need to find u_1, u_2, u_4 only. We assume roughly $u_2 = 3$ [since u_2 is at ½ distance from the value $u=2$]
[By interpolation technique $u_2 = 2+\frac{1}{3}(5-2)=3$]

Rough values:

$$u_1 = \frac{1}{4}[1+1+2u_3] = 2$$

$$u_2 = 3, \quad u_4 = \frac{1}{4}[5+5+2u_2] = 4$$

1st iteration:

$$u_1 = \frac{1}{4}[2+2u_2] = 2$$

$$u_2 = \frac{1}{4}[2+u_4+4+u_1] = \frac{12}{4} = 3$$

$$u_4 = \frac{1}{4}[5+5+2u_2] = 4$$

Since the rough values & 1st iteration values are identical, we get $u_1=2$, $u_2=3$, $u_4=4$ & $u_3=3$

Two dimensional poisson equation

Solution of poisson equation:

An equation of the form $\nabla^2 u = f(x,y)$

(ie) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \quad \text{--- (1)}$$

is called poisson's equation. where $f(x,y)$ is a fun. of x & y only.

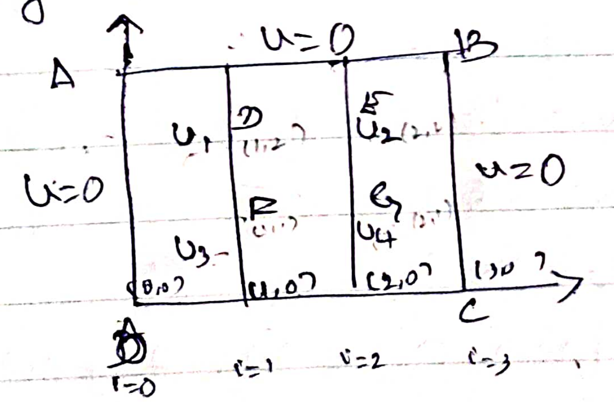
We will solve the above equation numerically over a square mesh, replacing the derivatives by difference quotients. Taking $x = ih$, $y = jk = jh$ the differential equation reduces to

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} = f(ih, jh)$$

(ie) $$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh) \quad \text{--- (2)}$$

By applying the above formula at each mesh point, we get a system of linear equation.

Eg 1) Solve $\nabla^2 u = -10(x^2+y^2+10)$ over the square mesh, with sides $x=0, y=0, x=3, y=3$ with $u=0$ on the boundary & mesh length 1 unit.



$f(x,y)$

Soln: Given P.D.E is $\nabla^2 u = -10(x^2+y^2+10)$ — ①

Given mesh length $\Delta x = h = 1$.

The formula is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)$$

$$\Rightarrow u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(i^2+j^2+10) \quad \text{— ②}$$

$\because h=1$

Applying the formula ② at D $(i=1, j=2)$

$$0+0+u_2+u_3-4u_1 = -10(1+4+10) = -150$$

$$\Rightarrow u_2+u_3-4u_1 = -150 \quad \text{— ③}$$

Applying the formula ② at E $(i=2, j=2)$

$$u_1+u_4-4u_2 = -180 \quad \text{— ④}$$

Applying the formula ② at F $(i=1, j=1)$

$$u_1+u_4-4u_3 = -120 \quad \text{— ⑤}$$

Applying ② at G, $(i=2, j=1)$

$$u_2+u_3-4u_4 = -10(2^2+1+10) = -150 \quad \text{— ⑥}$$

We can solve the equation ③, ④, ⑤, ⑥ either by direct elimination (or) by Gauss-Seidal method.

Method I:

⑤ - ④ gives (Eliminat u_1)

$$4(u_2-u_3) = 60$$

$$u_2-u_3 = 15 \quad \text{— ⑦}$$

Eliminate u_1 from ③ & ④, 3+4 (4) gives

$$-15u_2 + u_3 + 4u_4 = -870 \quad \text{--- (8)}$$

Adding (6) & (8) $\quad -7u_2 + u_3 = -510 \quad$ --- (9)

From (7), (9) adding $u_2 = 82.5$

Using (7), $u_3 = u_2 - 15 = 82.5 - 15 = 67.5$

Put in (8), $4u_1 = 300$, $\therefore u_1 = 75$

$4u_4 = 150 + 150 \Rightarrow u_4 = 75$

$\therefore u_1 = u_4 = 75$, $u_2 = 82.5$, $u_3 = 67.5$

Method: 2:

We can use Gauss-seidal method to solve

$$u_1 = \frac{1}{4}(150 + u_2 + u_3)$$

$$u_2 = \frac{1}{4}(2u_1 + 180)$$

$$u_3 = \frac{1}{4}(2u_1 + 120)$$

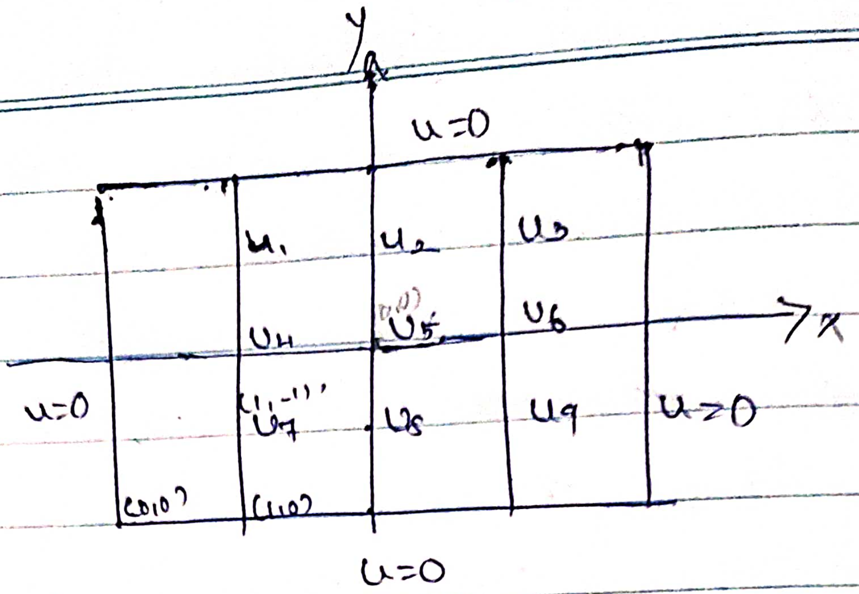
	1	2	3	4	5	6	7	8	9	10
$u_4 = u_1$	-	37.5	65.56	72.64	74.41	74.85	74.96	74.99	75	75
u_2	0	63.75	77.79	81.32	82.21	82.43	82.48	82.5	82.5	82.5
u_3	0	48.75	62.78	66.32	67.21	67.43	67.48	67.5	67.5	67.5

We get the values after 9 iterations as

$u_1 = 75 = u_4$, $u_2 = 82.5$, $u_3 = 67.5$.

2) Solve $\nabla^2 u = 8x^2y^2$ for square mesh given $u = 0$ on the 4 boundaries dividing the square in to 16 sub-squares of length 1 unit.

Soln:



Given P.D.E is $\nabla^2 u = 8x^2y^2$

Given mesh length $\Delta x = h = 1$. Take the coordinates system with origin at the centre of the square.

Since the P.D.E and boundary conditions are symmetrical about x, y axes & $y = x$ we have $u_1 = u_3 = u_7 = u_9$,

$u_2 = u_4 = u_6 = u_8$.

$\therefore$ we need to find u_1, u_2, u_5 only

As per the five pt. formula [here $f(x,y) = 8x^2y^2$]

$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)$

$= f(i,j)$

$= 8(i^2)(j^2)$ —— (1)

At $(i=-1, j=1)$, we have

$u_2 + u_4 - 4u_1 = 8(-1)^2$

(or) $u_2 - 2u_1 = 4$ —— (2)

At $(x=0, y=1)$

$$u_1 + u_3 + u_5 + 0 - 4u_2 = 0$$

$$2u_1 - 4u_2 + u_5 = 0 \quad \text{——} \quad (3)$$

At $(x=0, y=0)$

$$u_2 + u_4 + u_6 + u_8 - 4u_5 = 0$$

$$4u_2 - 4u_5 = 0$$

$$u_2 - u_5 = 0 \quad \text{——} \quad (4)$$

From (2) $\quad u_1 = \frac{1}{2}(u_2 - 4)$

From (4) $\quad u_5 = u_2$

Using in (3) $\quad u_2 - 4 - 4u_2 + u_2 = 0$

$$\therefore u_2 = -2.$$

$$\therefore u_5 = -2, \; u_1 = -3$$

$$\therefore u_1 = -3, \; u_2 = -2 = u_5$$

Hence $\quad u_1 = u_3 = u_7 = u_9 = -3$

$$u_2 = u_4 = u_6 = u_8 = u_5 = -2.$$