PROBABILITY AND QUEUEING THEORY

UNIT - I

RANDOM VARIABLES

RANDOM VARIABLE:

A random variable is a rule that assigns a numerical value to each possible outcome of an experiment.

Eg: Consider an experiment of tossing an unbiased coin twice. The outcome of the experiment are {HH, HT, TH, TT}. Let X denote the number of heads turning up. Then X has the values 2, 1, 1, 0. Here X is a random variable which assigns a real number to every outcome of a random experiment.

Def: Let S be a sample space associated with an experiment E. A function $X : S \to R$ which assigns to each element $s \in S$ a unique real number $X(s) \in R$ is called a random variable.

Types of Random Variable:

1. Discrete Random variables.
2. Continuous Random variables.

Discrete Random Variable:

Let x be a random variable. If the number of possible values of x is finite (or) Countably infinite then x is called a discrete random variable.

Eg: Let x represent the sum of numbers on the 2 dice, when two dice are thrown. In this case the random variable x takes the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. So x is discrete random variable.

Continuous Random Variable:

If x is a Random variable which can take all values (ie infinite number of Values) in an interval, then x is called a continuous Random Variable.

Eg: Age, height, weight are continuous R.V's

Discrete Random Variable	Continuous Random Variable.
P.m.f [Probability mass function] If x is a discrete R.V which takes the values $x_1, x_2, x_3 \ldots$ Such that $p[x = x_i] = p_i$, then p_i is Called the pmf of x provided it Satisfies the following Condition (i) $p[x_i] \geq 0, \forall i$ & (ii) $\sum_{i=1}^{\infty} p(x_i) = 1$.	**Pdf [probability density function]** Let x be a Continuous RV the function $f(x)$ is called the probability density function (pdf) of x, if it satisfies the following condition. (i) $f(x) \geq 0$, $\forall x$ in R_x (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (iii) $f(x) = 0$ if x is not in R_x Note: $p(a \leq x \leq b)$ (or) $p(a < x < b)$ is defined as $p(a \leq x \leq b) = \int_a^b f(x)\,dx$.

cdf [Cumulative distribution function]	cdf [Cumulative distribution function]
$P(X \le x) = F(x) = \sum_j p_j$	$F(x) = P[-\infty < X \le x] = \int_{-\infty}^{x} f(x)\,dx$
Properties of cdf	**Properties of cdf:**
1. $F(x)$ is non decreasing function of x. If $a \le b$, then $F(a) \le F(b)$	1. $F(x)$ is non-decreasing fun of x. If $a < b$. then $F(a) \le F(b)$.
2. $F[-\infty] = 0$ & $F[\infty] = 1$	2. $F[-\infty] = 0$ & $F[\infty] = 1$
3. If X is discrete R.V taking the values $x_1, x_2, \ldots$ where $x_1 < x_2 < \ldots < x_{i-1} < x_i \ldots$ then $P[X = x_i] = F[x_i] - F[x_{i-1}]$	3. If X is a Continuous RV then $\frac{d}{dx}[F(x)] = f(x)$, at all points where $F(x)$ is differentrable.

Problems based on discrete R.Vs:-

1. Consider a random experiment of tossing three times a fair coin. Let X denote the number of heads and Y denote the number of consecutive heads. Find

(i) the Probability distribution of X & Y.

(ii) the distribution function of X.

(iii) the Probability distribution of $X+Y$ & XY.

Soln:

The Sample Space S of the random experiment is $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Each element of S occurs with Probability $1/8$.

The values of $X, Y, X+Y$ and XY for outcome are tabulated as below:

Event	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
X	3	2	2	1	2	1	1	0
Y	3	2	0	0	2	0	0	0
$X+Y$	6	4	2	1	4	1	1	0
XY	9	4	0	0	4	0	0	0

(i) Probability distribution of X:

Values of X	0	1	2	3
$p(x)$	$1/8$	$3/8$	$3/8$	$1/8$

(ii) Probability distribution of Y:

Values of Y	0	2	3
$p(y)$	$5/8$	$2/8$	$1/8$

(iii) Distribution fun. of X:

X	$(-\infty,0)$	$[0,1)$	$[1,2)$	$[2,3)$	$[3,\infty)$
$F(x)$	0	$1/8$	$4/8$	$7/8$	1

(or) $$F(x) = \begin{cases} 0 & \text{if } x<0 \\ 1/8 & \text{if } 0\le x<1 \\ 4/8 & \text{if } 1\le x<2 \\ 7/8 & \text{if } 2\le x<3 \\ 1 & \text{if } x\ge 3 \end{cases}$$

(iv) Probability distribution of $Z=X+Y$ & $W=XY$

Values of Z	0	1	2	4	6
$p(z)$	$1/8$	$3/8$	$1/8$	$2/8$	$1/8$

Values of W	0	4	9
$p(w)$	$5/8$	$2/8$	$1/8$

2. A random variable X has the following probability function

X	0	1	2	3	4	5	6	7
$p(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

a) Find k

b) Evaluate $p[X<6]$, $p[X\geq 6)$ & $p(0<X<4)$

c) If $p[X \leq c] > 1/2$ find the minimum value of c.

d) Evaluate $p[1.5 < X < 4.5 \mid X>2]$

e) Find $p[X<2]$, $p[X>3]$, $p[1<X<5]$.

~~f) Find the distribution of X~~

Solu:

a) We know that $\sum p(x) = 1$

(ie) $0+k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$

(ie) $10k^2+9k-1=0$

$\Rightarrow 10k^2-k+10k-1=0$

$k(10k-1)+1(10k-1)=0$

$\Rightarrow (10k-1)(k+1)=0$.

$\therefore k=1/10$ (or) $k=-1$

$k=-1$ is not possible each probability >0

Hence $k=1/10$.

b) $p[X<6] = p[X=0]+p[X=1]+ \dots p[X=5]$

$= 0+k+2k+2k+3k+k^2$

$= k^2+8k = \frac{1}{100}+\frac{8}{10} = \frac{81}{100}$

$p[X\geq 6] = 1-p[X<6]$

$= 1-81/100 = 19/100$

$$p[0 < x < 4] = p[x=1] + p[x=2] + p[x=3]$$
$$= k + 2k + 2k$$
$$= 5k = 5\left[\frac{1}{10}\right] = \frac{1}{2}$$

c)

x_i	$p(x_i)$	$F(x) = p[X \le x]$
0	0	$F(0) = p(0) = 0$
1	$1/10$	$F(1) = F(0) + p(1) = 0 + \frac{1}{10} = \frac{1}{10}$
2	$2/10$	$F(2) = F(1) + p(2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$
3	$2/10$	$F(3) = F(2) + p(3) = \frac{3}{10} + \frac{2}{10} = \frac{5}{10}$
4	$3/10$	$F(4) = F(3) + p(4) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10}$
5	$\frac{1}{100}$	$F(5) = F(4) + p(5) = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$
6	$\frac{2}{100}$	$F(6) = F(5) + p(6) = \frac{81}{100} + \frac{2}{100} = \frac{83}{100}$
7	$\frac{7}{100} + \frac{1}{100} = \frac{17}{100}$	$F(7) = F(6) + p(7) = \frac{83}{100} + \frac{17}{100} = \frac{100}{100} = 1$

$\therefore$ The minimum value of $c = 4$ $\left\{\because p[X \le c] > \frac{1}{2}\right\}$

d) $p[1.5 < x < 4.5 \mid X > 2]$

$$= \frac{p[(1.5 < x < 4.5) \cap (X > 2)]}{p[X > 2]}$$

$\left[\because p(A/B) = \frac{p(A \cap B)}{p(B)}\right.$ Conditional prob.$\left.\right]$

$$= \frac{p[2 < x < 4.5]}{1 - p[X \le 2]}$$

$$= \frac{p(3) + p(4)}{1 - [p(0) + p(1) + p(2)]}$$

$$= \frac{\frac{2}{10} + \frac{3}{10}}{1 - \left[0 + \frac{1}{10} + \frac{2}{10}\right]} = \frac{\frac{5}{10}}{1 - 3/10} = \frac{5}{10} \cdot \frac{10}{7}$$

$$= 5/7$$

(e) i) $p[x<2] = p[x=0] + p[x=1]$

$= 0 + k = 1/10$

(ii) $p[x>3] = 1 - p[x \le 3]$

$= 1 - \{p[x=0] + p[x=1] + \cdots p[x=3]\}$

$= 1 - [0 + k + 2k + 2k]$

$= 1 - 5k = 1 - 5/10 = 1 - 1/2 = 1/2$

(iii) $p[1<x<5) = p[x=2] + p[x=3] + p[x=4]$

$= 2k + 2k + 3k = 7k = 7/10$

3. The Probability function of an infinite discrete distribution is given by $p[x=j] = \frac{1}{2^j}$ $[j = 1, 2 \ldots]$ Find (i) Mean of x (ii) p[x is even] (iii) $p[x \ge 5]$ (iv) p[x is divisible by 3]

Solu:-

$p[x=j] = \frac{1}{2^j}, \ j = 1, 2, 3 \ldots$

(i) Mean of $x = \Sigma x p(x)$

$= \sum_{j=1}^{\infty} j\left(\frac{1}{2^j}\right)$

$= \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots$

$= \frac{1}{2}[1 + 2(1/2) + 3(1/2^2) + 4(1/2^3) + \cdots]$

$= 1/2 [1 - 1/2]^{-2} = \left(\frac{1}{2}\right)^{-1} = 2$

(ii) p[x is even] $= p[x=2] + p[x=4] + p[x=6] + \cdots$

$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots$

$= \frac{1}{2^2}[1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots]$

$= \cancel{1/4}\left[\frac{1/4}{1 - 1/4}\right] = 1/3$

(iii) $p[x>5] = p[x=6] + p[x=7] + \dots$

$= \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \dots$

$= a + ar + ar^2 \dots$ $\quad \therefore a = \frac{1}{2^6}, r = \frac{1}{2}$

$= \frac{a}{1-r}$

$= \frac{1/64}{1-1/2}$

$= 1/32$ //

(iv) $p[x$ is divisible by $3] = p[x=3] + p[x=6] + \dots$

$= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \dots$

$= \frac{1}{2^3}[1 + \frac{1}{2^3} + \frac{1}{2^6} + \dots]$

$= \boxed{\frac{a}{1-r}}$ $\quad a = (\frac{1}{2})^3 = 1/8$

$= 1/8 / 1 - 1/8$

$= 1/7$ //

Problems based on continuous R.Vs:

1. If the density function of a continuous R.V X is given by.

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

(i) find the value of 'a'

(ii) find the cdf of X

(iii) Compute $p[X \le 1.5]$ & $p[X > 1.5]$

Soln:

(i) Since $f(x)$ is a pdf,

$$\int f(x)\,dx = 1$$

(ie) $\int_0^3 f(x)\,dx = 1$

(ie) $\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$

$\Rightarrow \left[\frac{ax^2}{2}\right]_0^1 + [ax]_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1$

$\frac{a}{2} + a - \frac{5a}{2} + 3a = 1$

$\Rightarrow 2a = 1$

$\therefore a = 1/2$

(ii) $F(x) = p[X \le x]$

$F(x) = 0$ when $x < 0$

$F(x) = \int_0^x \frac{x}{2}\,dx = \frac{x^2}{4}$ when $0 \le x < 1$

(iii) If $1 \le x \le 2$ then $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} ax\,dx + \int_{1}^{x} a\,dx$$

$$= a\left[\frac{x^2}{2}\right]_0^1 + a[x]_1^x$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}[x]_1^x \qquad \left[\because a = \frac{1}{2}\right]$$

$$= \frac{1}{4}[x^2]_0^1 + \frac{1}{2}[x-1]$$

$$= \frac{1}{4} + \frac{1}{2}(x-1)$$

$$= \frac{1}{4} + \frac{x}{2} - \frac{1}{2} = \frac{x}{2} - \frac{1}{4}$$

(iv) If $2 \le x \le 3$ then $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^x (3a - ax)\,dx$$

$$= \frac{1}{2}\int_0^1 x\,dx + \frac{1}{2}\int_1^2 dx + \int_2^x \left[\frac{3}{2} - \frac{x}{2}\right]dx \qquad [\because a = 1/2]$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}[x]_1^2 + \left[\frac{3}{2}x - \frac{1}{2}\frac{x^2}{2}\right]_2^x$$

$$= \frac{1}{4}[x^2]_0^1 + \frac{1}{2}[2-1] + \left[\frac{3}{2}x - \frac{x^2}{4}\right]_2^x$$

$$= \frac{1}{4}[1-0] + \frac{1}{2} + \left[\frac{3}{2}x - \frac{x^2}{4}\right] - (3-1)$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{3}{2}x - \frac{x^2}{4} - 2 = \frac{3}{2}x - \frac{x^2}{4} - \frac{5}{4}$$

(v) If $x > 3$ then $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{2}\,dx + \int_2^3 \left(\frac{3}{2} - \frac{x}{2}\right)dx - 0$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}[x]_1^2 + \left[\frac{3}{2}x - \frac{x^2}{4}\right]_2^3$$

$$= \frac{1}{4}[x^2]_0^1 + \frac{1}{2}[2-1] + \left[\frac{9}{2} - \frac{9}{4}\right] - (3-1)$$

$$= \frac{1}{4}(1) + \frac{1}{2} + \frac{9}{4} - 2 = \frac{10}{4} - \frac{3}{2} = 1$$

(iii) $P[X>1.5] = \int_{1.5}^{3} f(x)\,dx$

$$= \int_{1.5}^{2} \frac{1}{2}\,dx + \int_{2}^{3} \left(\frac{3}{2} - \frac{x}{2}\right) dx$$

$$= \frac{1}{2}[x]_{1.5}^{2} + \left[\frac{3}{2}x - \frac{x^2}{4}\right]_2^3$$

$$= \frac{1}{2}[2-1.5] + \frac{1}{4}$$

$$= \frac{1}{2}$$

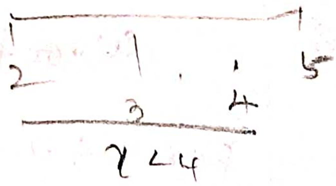
$\therefore P[X \le 1.5] = 1 - P[X>1.5]$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$ //

2) A continuous R.V X that can assume any value b/w $x=2$ & $x=5$ has the density function given by $f(x) = k(1+x)$. Find $P[X<4]$ & $P[3<X<4]$.

Solu:

To find k:



$$\int f(x)\,dx = 1$$

$$k\int_2^5 (1+x)\,dx = 1$$

$$\Rightarrow k\left[\frac{(1+x)^2}{2}\right]_2^5 = 1$$

$$\frac{k}{2}[36-9] = 1 \Rightarrow k = 2/27$$

$$P[X<4] = P[2<X<4] = \frac{2}{27}\int_2^4 (1+x)\,dx$$

$$= \frac{2}{27}\left[\frac{(1+x)^2}{2}\right]_2^4$$

$$= 16/27$$

$$P[3<X<4] = \frac{2}{27}\int_3^4 (1+x)\,dx = \frac{1}{27}\left[(1+x)^2\right]_3^4$$

$$= 9/27$$ //

3. A continuous R.V x has a pdf $f(x) = K\cancel{x}e^{-|x|}, \cancel{x \geq 0}$.
Find k, ~~mean~~ and ~~variance~~ the $F(x)$

$f(x) : Ke^{-|x|}, x > 0$

Solv:

Given $f(x)$ is a pdf

$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_{-\infty}^{\infty} k\,e^{-|x|}\,dx = 1$

$f(-x) = f(x)$ even fn

$\int_{-a}^{a} \underbrace{f(x)}_{\text{even}}\,dx = 2\int_0^a f(x)\,dx$

$\Rightarrow 2\int_0^{\infty} ke^{-x}\,dx = 1$

$2k\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$

$\left[e^{-\infty} = 0\right.$

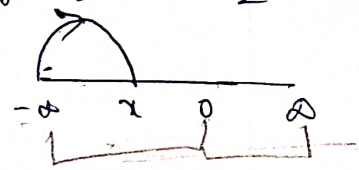
$2k\left[\left(\frac{0}{-1}\right) - \left(\frac{1}{-1}\right)\right] = 1$

$2k[0+1] = 1;\ 2k = 1;\ k = 1/2$

$F(x) = \int_{-\infty}^{x} f(x)\,dx$

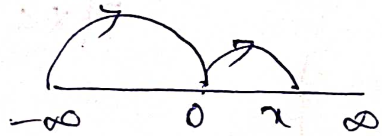
Given $f(x) = k\,e^{-|x|} = \begin{cases} ke^{x}; & -\infty < x < 0 \\ k\,e^{-x}; & 0 < x < \infty \end{cases} = \begin{cases} \frac{1}{2}e^{x}; & -\infty < x < 0 \\ \frac{1}{2}e^{-x}; & 0 < x < \infty \end{cases}$

For $x \leq 0$ $F(x) = \int_{-\infty}^{x} \frac{1}{2}e^{x}\,dx = \frac{1}{2}\left[e^{x}\right]_{-\infty}^{x} = \frac{1}{2}\left[e^{x} - 0\right] = \frac{1}{2}e^{x}.$



For $x > 0$ $F(x) = \int_{-\infty}^{0} \frac{1}{2}e^{x}\,dx + \int_0^{x} \frac{1}{2}e^{-x}\,dx$

$= \frac{1}{2}\left[e^{x}\right]_{-\infty}^{0} + \frac{1}{2}\left[\frac{e^{-x}}{-1}\right]_0^{x}$



$= \frac{1}{2}[1-0] + \frac{1}{2}\left[\frac{e^{-x}}{-1} - \left(\frac{1}{-1}\right)\right]$

$= \frac{1}{2} + \frac{1}{2}\left[-e^{-x} + 1\right]$

$= \frac{1}{2} + \frac{1}{2}\left[1 - e^{-x}\right]$

$= \frac{1}{2}\left[2 - e^{-x}\right]$

4. A continuous random variable X has the distribution function $F(x) = \begin{cases} 0 & ; x < 1 \\ k(x-1)^4 & : 1 \le x \le 3 \\ & x > 3 \end{cases}$

find k, probability density function $f(x)$, $p[X<2]$.

Soln:

W.K.T $p[X \le x] = F[x]$

$$f(x) = \frac{d}{dx} F(x)]$$

$$= \frac{d}{dx}[k(x-1)^4]$$

$$= k4(x-1)^3$$

pdf: $f(x) = \begin{cases} 0 & : x \le 1 \\ 4k(x-1)^3 & : 1 < x \le 3 \\ 0 & x > 3 \end{cases}$

To find k:

W.K.T $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_1^3 4k(x-1)^3 dx = 1$

$$4k\left[\frac{(x-1)^4}{4}\right]_1^3 = 1$$

$$k[(x-1)^4]_1^3 = 1$$

$$\Rightarrow k[(3-1)^4 - (1-1)^4] = 1$$

$$\Rightarrow k[2^4 - 0] = 1$$

$$\Rightarrow 16k = 1$$

$$\Rightarrow k = 1/16$$

$$p[X<2] = F[2] = k[2-1]^4$$

$$= \frac{1}{16}(1)^4$$

$$= 1/16$$

Moments :-

Mathematical Expectation :

Let x be a random variable with pdf (or pmf) $f(x)$. Then its mathematical expectation, denoted by $E(x)$ is given by

$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$ [for continuous R.V]

$E(X) = \sum x f(x)$ [for discrete R.V]

Moments: [Discrete case]

Let x be discrete R.V ~~taking~~ with pmf $p(x)$ then rth moment about the origin is

μ_r' (about the origin) (or) raw moments $= \sum x^r p(x)$

and μ_r' (about any point $x = A$) $= \sum (x-A)^r p(x)$

& μ_r (about mean) (ie) [central moments] $\} = \sum (x - \text{mean})^r p(x)$

[mean $= \bar{x}$]

Moments [continuous case]

If X is a continuous R.V with pdf $f(x)$ then defined in the interval (a,b)

μ_r' (about origin) (or) raw moments $= \int_a^b x^r f(x)\, dx$

μ_r' (about a point A) $= \int_a^b (x-A)^r f(x)\, dx$

μ_r (about the mean) (or) (central moments) $= \int_a^b (x-\bar{x})^r f(x)\, dx$

~~Moments Generating function~~:

Relation between μ_r' & μ_r

$$\mu_1' = \text{mean } \bar{x}$$

$$\mu_2 = \mu_2' - (\mu_1')^2 \quad [\text{variance}]$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

Note :

(1) $\mu_1 = 0$ (always)

(2) $E[ax+b] = aE(x) + b$

(3) $E[\phi(x)+a] = E[\phi(x)] + a$

(4) $\text{var}(ax+b) = a^2\,\text{var}(x)$

(5) $\text{var}(x+k) = \text{var}(x)$

$E(ax+b) = E(ax) + E(b$
$= aE(x) + b$

Problems based on Moments [Discrete Case]

1. The monthly demand for Allwyn watches is known to have the following probability distribution.

Demand	1	2	3	4	5	6	7	8
Probability	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Find the expected demand for watches. Also compute the variance.

Soln:

x_i	1	2	3	4	5	6	7	8
$p(x_i)$	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Let x be the R.v denoting the monthly demand for Allwyn watches.

$$E(x) = \sum_{i=1}^{8} x_i p(x_i)$$

$$= 1(0.08) + 2(0.12) + 3(0.19) + 4(0.24) + 5(0.16) + 6(0.10) + 7(0.07) + 8(0.04)$$

$$= 4.06$$

$$E(x^2) = \sum_{i=1}^{8} x_i^2 p(x_i)$$

$$= 1(0.08) + 4(0.12) + 9(0.19) + 16(0.24) + 25(0.16) + 36(0.10) + 49(0.07) + 64(0.04)$$

$$= 19.7$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= 19.7 - (4.06)^2$$

$$= 19.7 - 16.48$$

$$= 3.22$$

Problems based on Moments [Continuous Case]

1. The density function of a random variable 'x' is given by $f(x) = kx(2-x)$, $0 \le x \le 2$. Find k, mean, variance and r^{th} moment.

Soln: Given $f(x) = kx(2-x)$, $0 \le x \le 2$

To find k:

$$\int f(x)\,dx = 1$$

$$\int_0^2 kx(2-x)\,dx = 1$$

$$k\int_0^2 (2x - x^2)\,dx = 1$$

$$k\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$k\left[(4 - 8/3) - (0-0)\right] = 1$$

$$k\left[\frac{12-8}{3}\right] = 1$$

$$\Rightarrow k = 3/4$$

To Find r^{th} moment: $E[x^r]$

$$E[x^r] = \int_0^2 x^r f(x)\,dx$$

$$= \frac{3}{4}\int_0^2 x^r(2x - x^2)\,dx$$

$$= \frac{3}{4}\int_0^2 (2x^{r+1} - x^{r+2})\,dx$$

$$= \frac{3}{4}\left[\frac{2x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2$$

$$= \frac{3}{4}\left[\left(\frac{2\,2^{r+2}}{r+2} - \frac{2^{r+3}}{r+3}\right) - (0-0)\right]$$

$$= \frac{3}{4}\left[\frac{2^{r+3}}{r+2} - \frac{2^{r+3}}{r+3} \right]$$

$$= \frac{3}{4} \cdot 2^{r+3} \left[\frac{1}{r+2} - \frac{1}{r+3} \right]$$

$$= \frac{3}{4} 2^{r} \cdot 2^{3} \left[\frac{\cancel{r}+3-\cancel{r}-2}{(r+2)(r+3)} \right]$$

$$= \frac{3}{\cancel{4}} 2^{r} \cdot \cancel{8}^{2} \left[\frac{1}{(r+2)(r+3)} \right]$$

$$E[x^r] = \frac{6 \cdot 2^r}{(r+2)(r+3)} \quad \text{——— (1)}$$

<u>To find mean $E[x]$ & $var(x)$</u>

put $r=1$ in (1) $E[x] = \frac{6 \cdot 2}{3 \cdot 4}$

$= 1$

put $r=2$ in (1) $E[x^2] = \frac{6 \cdot 2^2}{4 \times 5}$

$= 6/5$

$\therefore$ Mean $E[x] = 1$

& $var(x) = E[x^2] - [E[x]]^2$

$= 6/5 - 1$

$= \frac{6-5}{5} = 1/5$ //

2. Find the r^{th} moment about the origin for the distribution with pdf $f(x) = kx^2 e^{-x}$, $0 < x < \infty$. Hence find the 1st four moments about mean.

Soln:

To find k: $\int f(x)\,dx = 1$

$$\Rightarrow k\int_0^\infty x^{2} e^{-x}\,dx = 1 \qquad \left[\because \int_0^\infty e^{-x} x^{n-1}\,dx = \Gamma n\right.$$

$$k\int_0^\infty x^{3-1} e^{-x}\,dx = 1$$

$$k\,\Gamma 3 = 1 \qquad \left.\Gamma n = (n-1)!\right]$$

$$\Rightarrow k(2) = 1 \qquad [\because \Gamma 3 = 2! = 2]$$

$$k = 1/2$$

$$\therefore f(x) = \tfrac{1}{2} x^2 e^{-x},\ 0 < x < \infty$$

r^{th} moment about origin is $\mu_r' = \int_0^\infty x^r f(x)\,dx$

$$\therefore \mu_r' = \frac{1}{2}\int_0^\infty x^{r+2} e^{-x}\,dx = \frac{1}{2}\int_0^\infty x^{r+3-1} e^{-x}\,dx$$

$$= \frac{\Gamma r+3}{2} = \frac{(r+2)!}{2},\quad r = 1, 2, 3 \ldots$$

Moments about origin $\mu_1' = 3$, $\mu_2' = 12$, $\mu_3' = 60$ & $\mu_4' = 360$

Moments about mean $\mu_1 = 0$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 3$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 60 - 3(12)(3) + 2(27) = 6$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= 360 - 720 + 648 - 243$$

$$= 45$$

Moment Generating function [MGF]

Moment generating function of a random Variable x about the origin is defined as

$$M_x(t) = E[e^{tx}] = \sum e^{tx} p(x), \text{ if } x \text{ is discrete}$$
$$= \int_{-\infty}^{\infty} e^{tx} f(x)dx, \text{ if } x \text{ is continuous}.$$

Ily MGF about mean μ is defined by

$$M_{x-\mu}(t) = E[e^{t(x-\mu)}] = \sum e^{t(x-\mu)} p(x), \text{ if } x \text{ is discrete}$$
$$= \int e^{t(x-\mu)} f(x)dx \text{ if } x \text{ is continuous}$$

1. Prove that the r^{th} moment of the R.V 'x' about origin $M_x(t) = \sum \frac{t^r}{r!} \mu_r'$.

Proof:

we know that

$$M_x(t) = E[e^{tx}]$$
$$= E\left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots \frac{(tx)^r}{r!} + \dots\right]$$
$$\left[\text{formula } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots\right]$$
$$= E[1] + \frac{E[tx]}{1!} + E\left[\frac{t^2x^2}{2!}\right] + \dots E\left[\frac{t^r x^r}{r!}\right] + \dots$$
$$= 1 + t\,E[x] + \frac{t^2}{2!}E[x^2] + \dots \frac{t^r}{r!}E[x^r] + \dots$$
$$M_x(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \dots \frac{t^r}{r!}\mu_r' + \dots$$
$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

Note: Thus r^{th} moment = coefficient of $\frac{t^r}{r!}$

Note: 2 $\mu_r' = \dfrac{d^r}{dt^r}\left[M_x(t)\right]_{t=0}$.

Properties of MGF:

$M_x(t) = E[e^{tx}]$

$M_{x-a}(t) = E[e^{t(x-a)}]$

1. $M_{x-a}(t) = e^{-at} M_x(t)$

Proof: $M_{x-a}(t) = E[e^{t(x-a)}]$

$= e^{-at} E[e^{tx}]$

$= e^{-at} M_x(t)$

2. Let $Y = ax+b$, where x is a R.V with moment generating fun. $M_x(t)$. Then $M_Y(t) = e^{bt} M_x(at)$.

Proof:

$M_Y(t) = E[e^{tY}]$

$= E[e^{t(ax+b)}]$

$= e^{bt} E[e^{x(at)}]$

$= e^{bt} M_x(at)$

3. If $M_x(t) = E[e^{tx}]$ then $M_{cx}(t) = M_x(ct)$

Solu: $M_{cx}(t) = E[e^{t(cx)}] = E[e^{(ct)x}] = M_x(ct)$

4. If X & Y are two independent RV, then $M_{X+Y}(t) = M_X(t).M_Y(t)$.

Proof: $M_{X+Y}(t) = E[e^{t(X+Y)}]$

$= E[e^{tX+tY}]$

$= E[e^{tX}.e^{tY}]$

$E[XY] = E[X].E[Y]$

X & Y

$= E[e^{tX}].E[e^{tY}]$ [$\because X$ & Y are independent]

$M_{X+Y}(t) = M_X(t).M_Y(t)$

Note: If $X_1, X_2, \ldots, X_n$ are n independent RVs then

$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t).M_{X_2}(t) \cdots M_{X_n}(t)$.

Problems based on MGF [Discrete case]

1. A random variable x has probability function $p(x) = \frac{1}{2^x}$; $x = 1, 2, 3 \dots$. Find the MGF, Mean & Variance.

Soln:

W.K.T $M_x(t) = \Sigma e^{tx} p(x)$ (for discrete R.V) $\boxed{M_x(t) = E[e^{tx}]}$

$$= \Sigma e^{tx} \frac{1}{2^x}$$

$$= \Sigma \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \dots$$

$$= \frac{e^t}{2}\left\{1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \dots\right\}$$

$$\therefore M_x(t) = \frac{e^t}{2} \cdot \frac{1}{1-e^t/2} = \frac{e^t}{2-e^t}$$

$$\mu_1' = M_x'(0) = \frac{d}{dt}\left[\frac{e^t}{2-e^t}\right]_{t=0}$$

$$= \left[\frac{(2-e^t)e^t - e^t(-e^t)}{(2-e^t)^2}\right]_{t=0}$$

$$= \left[\frac{2e^t - \cancel{e^{2t}} + \cancel{e^{2t}}}{(2-e^t)^2}\right]_{t=0}$$

$$\mu_1' = 2$$

$$\mu_2' = \frac{d^2}{dt^2}[M_x(t)]_{t=0}$$

$$= \frac{(2-e^t)^2(2e^t) - 4e^t(2-e^t)(-e^t)}{(2-e^t)^4}$$

$$= \left[\frac{(2-e^t)2e^t + 4e^{2t}}{(2-e^t)^3}\right]_{t=0}$$

$$= 2+4 = 6$$

$\therefore$ Var $= \mu_2' - (\mu_1')^2$

$= 6 - 4 = 2$

$\therefore$ Mean $= 2$ & Variance $= 2$.

2. Find the MGF for the distribution with

$$p(x) = \begin{cases} 2/3, & \text{at } x = 1 \\ 1/3, & \text{at } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

Also find mean & Variance.

Soln:

X takes the values 1 & 2 with probabilities $2/3$ & $1/3$ respectively.

$$M_X(t) = \sum_{x=1}^{2} e^{tx} p(x)$$

$$= \frac{2}{3} e^t + \frac{1}{3} e^{2t}$$

$\therefore M_X(t) = \frac{2}{3} e^t + \frac{1}{3} e^{2t}$

$M_X'(t) = 2/3\, e^t + 2/3\, e^{2t}$ —— (1)

$M_X''(t) = 2/3\, e^t + \frac{4}{3} e^{2t}$ —— (2)

(1) $\Rightarrow \mu_1' = M_X'(t)\big|_{t=0} = 4/3$

(2) $\Rightarrow \mu_2' = M_X''(t)\big|_{t=0} = 6/3 = 2$

Hence mean $\mu_1' = 4/3$ &

Variance $= \mu_2' - \mu_1^2$

$= 2 - 16/9 = 2/9$

Problems based on MGF [continuous case]

1. Find the MGF of a random variable having the pdf $f(x) = \begin{cases} 1/3, & -1 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$

Soln:

$$M_X(t) = E[e^{tX}] = \int e^{tx} f(x)\,dx$$

$$= \int_{-1}^{2} e^{tx} \frac{1}{3}\,dx = \frac{1}{3}\left[\frac{e^{tx}}{t}\right]_{-1}^{2}$$

$$= \frac{1}{3t}\left[e^{2t} - e^{-t}\right] \quad t \neq 0.$$

For $t=0$ $M_X(0) = \lim_{t \to 0} \frac{e^{2t} - e^{-t}}{3t} = \frac{0}{0}$

$$= \lim_{t \to 0} \frac{2e^{2t} + e^{-t}}{3} = 3/3 = 1$$

$$\therefore M_X(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

2. Find the MGF of a r.v X whose pdf is defined by $f(x) = \begin{cases} x, & \text{for } 0 \le x \le 1 \\ 2-x, & \text{for } 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$ Hence find mean & var. of X

Soln:

$$M_X(t) = \int e^{tx} f(x)\,dx$$

$$= \int_0^1 x e^{tx}\,dx + \int_1^2 (2-x) e^{tx}\,dx$$

$$= \left[x\left[\frac{e^{tx}}{t}\right] - \frac{e^{tx}}{t^2}\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} + \frac{e^{tx}}{t^2}\right]_1^2$$

$$= \left[\cancel{\frac{e^t}{t}} - \frac{e^t}{t^2} - 0 + \frac{1}{t^2}\right] + \left[0 + \frac{e^{2t}}{t^2} - \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2}\right]$$

$$= \left[\frac{1 - 2e^t + e^{2t}}{t^2}\right] = \left(\frac{1-e^t}{t}\right)^2$$

To find mean & variance.

$$M_x(t) = \left[\frac{1-e^t}{t}\right]^2 \qquad e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots$$

$$= \left\{\frac{1-\left[1+\frac{t}{1!}+\frac{t^2}{2!}+\cdots\right]}{t}\right\}^2$$

$$= \left\{\frac{-\left[\frac{t}{1}+\frac{t^2}{2}+\frac{t^3}{6}+\cdots\right]}{t}\right\}^2$$

$$M_x(t) = \left[1+\frac{t}{2}+\frac{t^2}{6}+\cdots\right]^2$$

$$\mu_1' = \left[\frac{d}{dt}M_x(t)\right]_{t=0} = M_x'(0)$$

$$M_x'(t) = 2\left(1+\frac{t}{2}+\frac{t^2}{6}+\cdots\right)\left[\frac{1}{2}+\frac{t}{3}+\cdots\right] \quad \text{—— (1)}$$

$$M_x'(0) = 2(1)(1/2) = 1 \Rightarrow \boxed{\text{Mean} = 1}$$

$$(1) \Rightarrow M_x''(t) = 2\left(1+\frac{t}{2}+\frac{t^2}{6}\cdots\right)\left(1/3+\cdots\right) + 2\left(\frac{1}{2}+\frac{t}{3}+\cdots\right)\left(1/2+t/3+\cdots\right)$$

$$M_x''(0) = 2[1][1/3] + 2(1/2)\cdot(1/2)$$

$$= 2/3 + 1/2 = \frac{4+3}{6} = 7/6$$

$$\text{Variance of } x = E[x^2] - [E(x)]^2$$

$$= \mu_2' - \mu_1^2$$

$$= 7/6 - 1 = 1/6$$

Hence $\boxed{\text{Mean} = 1}$ & $\boxed{\text{Variance} = 1/6}$

Standard distributions

Discrete distributions	Continuous distribution
1. Binomial distribution	1. Uniform (or) Rectangular distribution
2. Poisson distribution	2. Exponential distribution
3. Geometric distribution	3. Gamma distribution
4. Negative Binomial distribution	4. Weibull distribution
	5. Normal distribution

Bernoulli Trial:

Each trial has two possible outcomes, generally called success and failure. Such a trial is known as Bernoulli trial.

The sample space for a Bernoulli trial is $S = \{s, f\}$.

Eg: 1. A toss of a single coin [head (or) tail]

2. The throw of a die (even (or) odd number)

Binomial Experiment:

An experiment consisting of a repeated number of Bernoulli trials is called Binomial experiment.

A Binomial experiment must possess the following properties

(i) There must be a fixed number of trials

(ii) All trials must have identical probabilities of success (p).

(iii) The trials must be independent of each other.

Binomial distribution:

A Discrete random variable X is said to follow binomial if it assumes only non-negative values & its Probability mass function is given by

$p(X=x) = nC_x\, p^x q^{n-x}, \quad x = 0, 1, 2 \ldots n \quad \& \quad q = 1-p.$

Notation:

$X \sim B(n, p)$, n & p are the parameters of Binomial distribution (ie) X follows Binomial distribution with parameters n & p.

Note:

$$\sum p(X=x) = \sum_{x=0}^{n} nC_x\, p^x q^{n-x} = (q+p)^n = 1 \qquad [\because p+q=1]$$

Binomial frequency distribution:

If we assume that n trials constitute a set and if we consider N sets, then the frequency function of the binomial distribution is given by

$$f(x) = N\, p(x)$$
$$= N\, nC_x\, p^x q^{n-x}, \quad x = 0, 1, 2 \ldots n$$

The expected frequencies of $0, 1, 2 \ldots n$ success are given by the successive terms of $N(q+p)^n$.

Mean & Variance of Binomial distribution

The pmf of Binomial distribution is

$$p(x) = nc_x\, p^x q^{n-x}$$

$$\text{Mean} = E(x) = \sum x\, p(x) = \mu_1'$$

$$= \sum_{x=0}^{n} x\, nc_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{\cancel{x}\, n(n-1)!}{\cancel{x}(x-1)!(n-x)!} p^x q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{n-x}$$

$$= np \sum (n-1)C_{x-1}\, p^{x-1} q^{n-x}$$

$$= np\,(q+p)^{n-1}$$

$$= np\,(1) = np$$

$$\Rightarrow \boxed{E(x) = np}$$

$$x^2 = x(x-1) + x$$
$$= x^2 - \cancel{x} + \cancel{x}$$

$$E[x^2] = \sum_{x=0}^{n} x^2\, nc_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} [x(x-1) + x]\, nc_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{x(x-1)\, n!}{x!(n-x)!} p^x q^{n-x} + \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=2}^{n} \frac{\cancel{x(x-1)}\, n(n-1)(n-2)!}{\cancel{x(x-1)}(x-2)!(n-x)!} p^x q^{n-x} + np \quad [\text{by } 1]$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 + np$$

$$Var(X) = E[X^2] - [E(X)]^2$$
$$= n(n-1)p^2 + np - (np)^2$$
$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$
$$= np[1-p]$$
$$= npq$$
$$\therefore \boxed{Var(X) = \mu_2 = npq}$$

~~Pro~~ MGF of a Binomial distribution about origin:

Solu:

pmf of Binomial distribution is

$$p(x) = nC_x\, p^x q^{n-x},\ x = 0, 1, 2 \ldots n$$

MGF of Binomial distribution about origin is

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{n} e^{tx} p(x)$$
$$= \sum e^{tx}\, nC_x\, p^x q^{n-x}$$
$$= \sum nC_x (pe^t)^x q^{n-x}$$
$$\therefore M_X(t) = (q + pe^t)^n$$

MGF of a Binomial distribution about the mean (np):

Solu.

$$M_X(t) = E[e^{t(X-np)}]$$
$$= E[e^{-npt} \cdot e^{tX}]$$
$$= e^{-npt} E[e^{tX}]$$
$$= e^{-npt} [q + pe^t]^n$$

Additive Property of Binomial distribution:

If $X \sim B(n_1, p)$ & $Y \sim B(n_2, p)$ then $X+Y \sim B(n_1+n_2, p)$ where X & Y are independent.

Proof:

Let $M_X(t) = [q+pe^t]^{n_1}$ & $M_Y(t) = (q+pe^t)^{n_2}$

Then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$ [$\because$ X & Y are independent]

$$= (q+pe^t)^{n_1} \cdot (q+pe^t)^{n_2}$$

$$= (q+pe^t)^{n_1+n_2}$$

This shows $X+Y$ is also binomial variate with parameters (n_1+n_2) & p

Problems:

1. Determine the binomial distribution for which the mean is 4 and the variance is 3.

Soln: Given $np = 4$ & $npq = 3$

$$\frac{npq}{np} = 3/4 \Rightarrow \boxed{q = 3/4}$$

& $p = 1 - q = 1 - 3/4 = 1/4$

$\therefore$ $\boxed{p = 1/4}$ & $q = 3/4$ Also $np = 4$

$n(1/4) = 4 \Rightarrow \boxed{n = 16}$

Hence the binomial distribution is given by

$$P[X=x] = nC_x \, p^x q^{n-x}$$

$$= 16C_x \left[\frac{1}{4}\right]^x \left[\frac{3}{4}\right]^{16-x}$$

$x = 0, 1, 2 \ldots 16.$

$10C_0 \; p^0 \; q^{10-0}$

$1 \times 1 \, (1/2)^{10}$

5

(2)

2. The mean of a Binomial distribution is 20 and S.D is 4. Determine the parameters of the distribution.

Solu:

Given Mean $= np = 20$

$S.D = \sqrt{npq} = 4$; $npq = 16$

$20q = 16$, $q = 16/20 = 4/5$

$p = 1-q = 1-4/5 = 1/5$, we have $np = 20$, $n(1/5) = 20$

$\Rightarrow n = 100$

The parameters of the binomial distributions are n, p. Here $n=100$, $p = 1/5$

3. If the mgf of a r.v X is of the form $(0.4e^t + 0.6)^8$, what is the mgf of $3X+2$. Evaluate $E(X)$.

Solu:

W.K.T $M_X(t) = (q+pe^t)^n$ —— (1)

Given $[0.4e^t + 0.6]^8 \Rightarrow p = 0.4$ & $q = 0.6$

$n = 8$.

$E[X] = np$

$= 8(0.4) = 3.2$

MGF of $3X+2$ is

$$M_{3X+2}(t) = E[e^{(3X+2)t}]$$

$$= e^{2t} E[e^{3Xt}]$$

$$= e^{2t}[0.4e^{3t} + 0.6]^8$$

(6)

4. 6 dice are thrown 729 times. How many times do you expect atleast three dice to show 5 (or) 6.

Solu:

The probability of getting 5 (or) 6 when a die is thrown $= \frac{2}{6} = \frac{1}{3}$.

$p = 1/3$; $q = 1-p = 1-1/3 = 2/3$ & $n = 6$

$p(X=x) = 6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$, $x = 0, 1, 2 \ldots 6$

∴ The probability of getting atleast 3 dice to show 5 (or) 6 is

$$p(X \geq 3) = p(X=3) + p(X=4) + p(X=5) + p(X=6)$$

$$= 6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^6 + 6C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 + 6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right) + 6C_6 \left(\frac{1}{3}\right)^6$$

$$= \frac{233}{729}.$$

In 729 times, when 6 dice are thrown,

$$= 729 \times \frac{233}{729}$$

$$= 233 \text{ times.}$$

In 233 times atleast 3 dice to show 5 (or) 6.

3

7

A

5. Out of 800 families with 4 children each, how many families would be expected to have (i) 2 boys & 2 girls (ii) at least 1 boy, (iii) at most 2 girls and (iv) children of both gender. Assume equal probabilities for boys & girls.

Soln:

Considering each child as a trial, $n=4$

Assuming that birth of a boy is a success. $p=1/2$ & $q=1/2$. Let x denote the number of successes (boys)

(i) $p(2 \text{ boys } \& \text{ 2 girls}) = p(x=2)$

$$= 4C_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^{4-2}$$

$$= 6 \times \left(\tfrac{1}{2}\right)^4 = 3/8$$

∴ No. of families having 2 boys & 2 girls

$$= N \cdot p(x=2)$$

$$= 800 \times 3/8 = 300$$

(ii) $p(\text{at least 1 boy}) = p(x \geq 1)$

$$= p(x=1) + p(x=2) + p(x=3) + p(x=4)$$

$$= 1 - p(x=0)$$

$$= 1 - 4C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^4$$

$$= 1 - \tfrac{1}{16} = \tfrac{15}{16}$$

∴ No. of families having atleast 1 boy

$$= 800 \times \frac{15}{16} = 750$$

(iii) $p(\text{at most 2 girls}) = p[\text{exactly 0 girl, 1 girl (or) 2 girls}]$

$$= p[x=4, x=3 \text{ (or) } x=2]$$

$$= [1 - (p(x=0) + p(x=1))]$$

$$= 1 - \{4C_0 (\tfrac{1}{2})^4 + 4C_1 (\tfrac{1}{2})^4\} = \frac{11}{16}$$

A

8

∴ No. of families having at most 2 girls $= 800 \times 11/16 = 550$

(iv) p(children of both gender) $= p(x=1)+p(x=2)+p(x=3)$

$= 1 - p(\text{children of the same gender})$

$= 1 - [p(\text{all are boys}) + p(\text{all are girls})]$

$= 1 - 4C_4(\frac{1}{2})^4 - 4C_0(\frac{1}{2})^4$

$= 1 - 1/8 = 7/8$

$\therefore$ No. of families having children of both gender

$= 800 \times 7/8 = 700$

6. An irregular 6-faced dice is such that probability that it gives 3 even numbers in 5 throws is twice the probability that it gives 2 even number in 5 throws. How many sets of exactly 5 trials can be expected to give no even number out of 2500 sets.

Soln:
Let p be the probability of getting an even no.
Let x denote the number of even number obtained in 5 trials.

Given $p(x=3) = 2p(x=2)$

$5C_3 p^3 q^2 = 2 \times 5C_2 p^2 q^3$

$5C_3 = \frac{5\times\cancel{4}^{2}\times\cancel{3}}{1\times\cancel{2}\times\cancel{3}} = 10$

$5C_2 = \frac{5\times 4^{2}}{1\times\cancel{2}} = 10$

5

$\Rightarrow p = 2q$

(ie) $p = 2(1-p)$

$3p = 2$ (or) $p = 2/3$ & $q = 1/3$

$\therefore$ Now p(getting no even number) $= p(x=0)$

$= 5C_0 p^0 q^5 = (1/3)^5 = 1/243$ (9)

$\therefore$ Number of sets having no success out of N sets $= N \times p(x=0) = 2500 \times 1/243 = 10$ nearly.

7. Two dice are thrown 120 times. Find the average number of times in which the number of the 1st die exceeds the number of the 2nd die.

Solu:

The number on the 1st die exceeds that on the second die in the following combinations.

(2,1) (3,1) (3,2) (4,1) (4,2) (4,3) (5,1) (5,2) (5,3) (5,4) (6,1) (6,2) (6,3) (6,4) (6,5)

$P\left(\begin{array}{l}\text{number in the 1st die exceeds the}\\ \text{number in the 2nd die}\end{array}\right) = \frac{15}{36} = \frac{5}{12}$.

Given that the trial is repeated 120 times.

$\Rightarrow n = 120$

$p = 5/12$

$\therefore E(X) = np = 120 \times 5/12 = 50$.

8. The probability of a bomb hitting a target is 1/5. Two bombs are enough to destroy a bridge. If sixs boms are aimed at the bridge, find the Probability that the bridge is destroyed.

Solu: Let p denotes the probability of a bomb hitting the target. Given $p = 1/5$

$q = 1 - p = 1 - 1/5 = 4/5$ & $n = 6$.

Let X denotes the no. of bombs to destroy a bridge

$X \sim B(n, p)$ (ie) $X \sim B(6, 1/5)$.

To find $p(X=2)$: $p(X=x) = nC_x p^x q^{n-x}$

$p(X=2) = 6C_2 (1/5)^2 (\frac{4}{5})^4$

$= 768/3125 = 0.2458$.

6

9. Comment on the following "The mean of a binomial distribution is 3 & variance is 4".

Solu: Mean = 3 & var = 4

(ie) $np = 3$ — (1) & $npq = 4$ — (2)

$\frac{(2)}{(1)}$ $q = \frac{4}{3} > 1$.

which is impossible. Since 'q' being a prob. of an event, $0 < q < 1$ but $q > 1$. ∴ The given data cannot determine a binomial distribution.

10. The prob. of a man hitting a target $1/4$.

(i) If he fires 7 times, what is the pro. of his hitting the target atleast twice?

(ii) How many times must he fire so that the prob. of his hitting the target atleast one is greater than $2/3$?

probability of hitting the target $p = 1/4$

$\therefore q = 1 - p = 1 - 1/4 = 3/4$ & $n = 7$.

(i) p[hitting the target atleast twice] $= p[X \geq 2]$

$= 1 - p(X < 2)$

$= 1 - [p(0) + p(1)]$

$= 1 - [(3/4)^7 + 7(1/4)(3/4)^6]$

$= 1 - \frac{10 \times 3^6}{4^7} = 0.551$ (app)

(ii) p[hitting the target atleast once] $= 1 - p[X = 0]$

$P(X \geq 1)$ $\quad$ $1 - P(X < 1)$

$1 - q^n$

W given $1 - q^n \geq 2/3$

$1 - (3/4)^n \geq 2/3$

$1 - 2/3 \geq (3/4)^n$

$\Rightarrow (3/4)^n \leq 1/3$

$\Rightarrow \boxed{n = 4}$

7

8

11. If X & Y are independent Binomial variate $B(5, 1/2)$ & $B(7, 1/2)$, find $P(X+Y=3)$

Soln:

If $X \sim B(n_1, p)$

$Y \sim B(n_2, p)$

$X+Y \sim B(n_1+n_2, p)$

$X \sim B(5, 1/2)$ $\quad n_1 = 5$

$Y \sim B(7, 1/2)$ $\quad n_2 = 7$ $\Rightarrow X+Y \sim B(12, 1/2)$

12-3

$\therefore P[X+Y=3] = 12C_3 (\frac{1}{2})^3 (1/2)^9 = \frac{55}{2^{10}}$.

8

Poisson distribution:

If X is a discrete RV that can assume the value 0,1,2.... Such that its Probability mass function is given by

$$P(X=x) = p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} : x = 0,1,2 \ldots\ldots, \lambda > 0$$

then X is said to follow a poisson distribution with parameter λ (or) Symbolically X is said to follow $p(\lambda)$.

Mean & variance of poisson distribution:

$$\text{Mean } E(X) = \sum x p(x)$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{\cancel{x} e^{-\lambda}\lambda^{x}}{\cancel{x}(x-1)!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty} \frac{\lambda \lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda}\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \qquad \left[e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} \ldots\right]$$

$$= \lambda e^{-\lambda}\left[e^{\lambda}\right]$$

$$\boxed{\text{Mean} = \lambda} \quad \text{--- } (1)$$

$$\text{Var}(X) = E(X^2) - \left[E(X)\right]^2$$

$$E(X^2) = \sum_{x=0}^{\infty} x^2 p(x)$$

$$= \sum_{x=0}^{\infty}\left[x(x-1)+x\right] p(x)$$

$$= \sum_{x=0}^{\infty} \frac{x(x-1)}{x!} e^{-\lambda}\lambda^{x} + \sum_{x=0}^{\infty} \frac{x}{x!} e^{-\lambda}\lambda^{x}$$

$$= \sum_{x=2}^{\infty} e^{-\lambda} \frac{\cancel{x(x-1)}\, \lambda^{x-2} \cdot \lambda^2}{\cancel{x(x-1)}\,(x-2)!} + \sum x\, p(x)$$

$$= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda \quad [\text{by } 1]$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda$$

$$= \lambda^2 + \lambda$$

$$\therefore \ var(x) = E(x^2) - [E(x)]^2$$

$$= \cancel{\lambda^2} + \lambda - \cancel{\lambda^2} = \lambda$$

$$\Rightarrow \boxed{var(x) = \lambda}$$

(x) Find MGF of poisson distribution about origin & hence find mean & variance.

→ put pdf.

$$M_x(t) = E[e^{tx}] = \sum e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$\therefore M_x(t) = e^{\lambda(e^t - 1)}$$

$$\text{Mean } E(x) = \frac{d}{dt} M_x(t) \Big|_{t=0}$$

$$= \frac{d}{dt} \left[e^{\lambda(e^t - 1)} \right] \Big|_{t=0}$$

$$= \frac{d}{dt} \left[e^{\lambda e^t} e^{-\lambda} \right] \Big|_{t=0}$$

$$= e^{-\lambda} \frac{d}{dt} \left[e^{\lambda e^t} \right] \Big|_{t=0}$$

$$= \left[e^{-\lambda} e^{\lambda e^t} \lambda e^t \right]_{t=0}$$

$$\frac{d}{dx} e^{ax} = a e^{ax}$$

$$= e^{-\lambda} e^{\lambda} \lambda = \lambda.$$

$$E[x^2] = \frac{d^2}{dt^2} M_X(t) \Big|_{t=0}$$

$$= \frac{d}{dt} M_X'(t) \Big|_{t=0}$$

$$= \frac{d}{dt} [\lambda e^{-\lambda} e^{\lambda e^t} . e^t] \Big|_{t=0}$$

$$= \lambda e^{-\lambda} \frac{d}{dt} [e^{\lambda e^t} . e^t] \Big|_{t=0}$$

$$= \lambda e^{-\lambda} [e^{\lambda e^t} . e^t + e^t \lambda e^t . e^{\lambda e^t}] \Big|_{t=0}$$

$$= \lambda e^{-\lambda} [e^{\lambda} + \lambda e^{\lambda}] = \lambda + \lambda^2$$

$$\therefore Var(X) = E[x^2] - E[X]^2$$

$$= \lambda + \lambda^2 - \lambda^2 = \lambda$$

$$\boxed{Var(X) = \lambda}$$

Poisson distribution as limiting form of Binomial distribution.

Poisson distribution is a limiting case of Binomial distribution under the following conditions.

(i) n, the number of trials is indefinitely large (ie) $n \to \infty$

(ii) p, the constant probability of success in each trial is very small (ie) $p \to 0$.

(iii) $np (= \lambda)$ is finite (or) $p = \lambda/n$ & $q = 1 - \lambda/n$ where λ is a positive real number.

If X is binomially distributed R.V with Parameter n & p, then $p(X=r) = nC_r p^r q^{n-r}$, $r = 0, 1, 2 \ldots n$

$$= \left[\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}\right] p^r (1-p)^{n-r}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!} \left(\frac{\lambda}{n}\right)^r (1-\lambda/n)^{n-r}$$

[on putting $p = \lambda/n$]

$$= \frac{\lambda^r}{r!}\left[\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{n^r}\right]\left[1-\lambda/n\right]^{n-r}$$

$$= \frac{\lambda^r}{r!}\left[1(1-1/n)(1-2/n)\ldots\ldots(1-(\tfrac{r-1}{n}))\right](1-\lambda/n)^n (1-\lambda/n)^{-r}$$

Taking limit on both sides.

$$\lim_{n\to\infty}\left[p(X=r)\right] = \frac{\lambda^r}{r!}\lim_{n\to\infty}\left[(1-1/n)(1-2/n)\ldots(1-(\tfrac{r-1}{n}))\right]$$

$$\lim_{n\to\infty}(1-\lambda/n)^n \cdot \lim_{n\to\infty}(1-\lambda/n)^{-r}$$

$$= \frac{\lambda^r}{r!}e^{-\lambda}$$

$\left[\because \lim_{n\to\infty}(1-\lambda/n) = 1\right.$

$\lim_{n\to\infty}(1-\lambda/n)^{-r} = 1$

$\left.\& \lim_{n\to\infty}(1-\lambda/n)^n = e^{-\lambda}\right]$

[But $p(X=r) = \dfrac{e^{-\lambda}\lambda^r}{r!}$

$r = 0, 1, \ldots\ldots$ is the pmf of the Poisson R.V]

Thus the limit of Binomial R.V is the Poisson R.V.

Problems:

1. If X is a poisson variate such that $p[X=2] = 9\,p[X=4] + 90\,p[X=6]$. Find (i) mean & $E(X^2)$ Also find $p[X \geq 2]$.

Soln:

W.k.T $p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$

Given $p[X=2] = 9\,p[X=4] + 90\,p[X=6]$

$$\Rightarrow \frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$1 = \frac{3}{4}\lambda^2 + \frac{\lambda^4}{4} \qquad \lambda^2 = x$$

$$\Rightarrow \lambda^4 + 3\lambda^2 - 4 = 0$$

$$x^2 + 3x - 4 = 0 \quad (x+4)(x-1) = 0$$

$$\therefore (\lambda^2+4)(\lambda^2-1) = 0$$

$$\lambda^2 = -4 \text{ (or) } \lambda^2 = 1$$

Hence $\lambda = 1$ [$\because \lambda^2$ cannot be $-$ve].

$\therefore$ Mean $\lambda = 1$ & $E[x^2] = \lambda^2 + \lambda = 2$ $\quad \lambda = 1$

$$p[X \geq 2] = 1 - p[X < 2]$$

$$= 1 - [p(0) + p(1)] = 1 - \left[\frac{e^{-1}1^0}{0!} + \frac{e^{-1}1^1}{1!}\right]$$

$$= 1 - e^{-1}[1+1] = 1 - 2/e$$

2. The no. of monthly breakdown of a computer is a random variable having a poisson distribution with mean equal to 1.8. Find the probability that this computer will fun. for a month.

(1) without a breakdown.

(2) with only one breakdown &

(3) with atleast one breakdown.

(9)

10

Solu:

Given mean $\lambda = 1.8$

Let x denotes the no. of breakdowns of a computer in a month.

$\therefore$ the Prob. distribution $p(x=x) = \frac{e^{-1.8}(1.8)^x}{x!}$

(a) p(without a breakdown) $= p(x=0) = \frac{e^{-1.8}(1.8)^0}{0!} = e^{-1.8}$

$= 0.1653.$

(b) p(with only one break down) $= p(x=1)$

$= \frac{e^{-1.8}(1.8)^1}{1!} = 0.2975$

(c) p(with atleast 1 breakdown) $= p(x \geq 1)$

$= 1 - p(x<1)$

$= 1 - p(x=0)$

$= 1 - 0.1653 = 0.8347$

3. A manufacturer of cotton pins that 5% of his Product is defective. If he sells pins in boxes of 100 and guarantees that not more than 4 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality?

Solu:

Let p be the probability of defective pins $= \frac{5}{100}$

$n = 100$

$\therefore \lambda = np = \frac{5}{100} \times 100 = 5$

Let x denote the No. of defectives in the box.

10

$\therefore$ Probability that a box will fail to meet the guaranteed quality $= p(x>4)$

$= \cancel{p = p}$

$= 1 - p(x \leq 4)$

$= 1 - [p(0) + p(1) + p(2) + p(3) + p(4)]$

$= 1 - e^{-5}\left[1 + 5 + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!}\right]$

$= 1 - e^{-5}\left[1 + 5 + \frac{25}{2} + \frac{125}{6} + \frac{625}{24}\right]$

$= 0.5620$

11

Geometric distribution:

If X is a discrete RV that can assume the values $1,2,3 \ldots \infty$ such that probability mass function is given by $P(X=x) = q^{x-1}p$, $x = 1,2 \ldots \infty$ where $p+q=1$. then X is said to follow a geometric distribution.

Find the MGF of a geometric RV with pmf $p(x) = pq^{x-1}$, $x = 1,2,3 \ldots$ & hence obtain its mean & variance.

Soln:

The pmf of Geometric distribution is $P(x) = q^{x-1}p$, $x = 1,2,3 \ldots$

$$M_X(t) = E[e^{tX}]$$

$$= \sum e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x$$

$$= \frac{p}{q} \{ qe^t + (qe^t)^2 + (qe^t)^3 + \ldots \}$$

$$= \frac{p\cancel{q}e^t}{\cancel{q}} \{ 1 + (qe^t) + (qe^t)^2 + \ldots]$$

$$= pe^t (1-qe^t)^{-1}$$

$$\therefore M_X(t) = \frac{pe^t}{1-qe^t}$$

(2)

To find mean & variance:

$$M_x'(t) = \frac{d}{dt}\left[\frac{pe^t}{1-qe^t}\right] = \frac{(1-qe^t)pe^t + pqe^{2t}}{(1-qe^t)^2}$$

$$= \frac{pe^t}{(1-qe^t)^2}$$

$$M_x''(t) = \frac{(1-qe^t)^2 pe^t + 2pe^t(1-qe^t)qe^t}{(1-qe^t)^4}$$

$$= \frac{(1-qe^t)pe^t + 2pe^t qe^t}{(1-qe^t)^3}$$

$$= \frac{(1+qe^t)pe^t}{(1-qe^t)^3}$$

$\therefore \mu_1'$ (about origin) $= M_x'(0) = \frac{p}{(1-q)^2} = \frac{1}{p}$

$$\mu_2' = M_x''(0) = \frac{p(1+q)}{(1-q)^3} = \frac{1+q}{p^2}$$

Hence mean $\mu_1' = \frac{1}{p}$ & Var $= \mu_2 = \mu_2' - \mu_1'^2$

$$= \frac{1+q}{p^2} - \frac{1}{p^2} = q/p^2.$$

Hence mean $= 1/p$

Var $= q/p^2$ //

Problems:

1. I) the probability that an applicant for a drivers licence will pass road test on any given trial is 0.8, what is the probability that he will finally pass test (a) on the 4th trial & (b) is fewer than 4 trials.

Soln:

Let X = No. of trials required to acheive the first success. Then X follows the geometric distribution given by

$P[X=x] = q^{x-1}p, \quad x = 1, 2, 3 \ldots$

Here $p = 0.8$ & $q = 0.2$

(a) $P[X=4] = (0.8)(0.2)^3 = 0.0064$

(b) $P[X<4] = \sum_{x=1}^{3} P[X=x] = P(1) + P(2) + P(3)$

$= \sum_{x=1}^{3} (0.8)(0.2)^{x-1} = 0.8(0.2)^0 + (0.8)(0.2)^1 + (0.8)(0.2)^2$

$= 0.8[1 + 0.2 + 0.04]$

$= 0.992$

2. A die is cast until 6 appears. What is the probability that it must cast more than five times?

Soln: Probability of getting six $= 1/6$

$\therefore p = 1/6$ & $q = 1 - 1/6 = 5/6$

12

(A)

1) Let x denote the no. of throws for getting the number. 6.

pmf of Geometric distribution is

$$P[X=x] = q^{x-1}p, \quad x=1,2,3 \ldots$$

To find $P[X>5] = 1 - P[X \le 5]$

$$= 1 - \sum_{x=1}^{5} \left(\frac{5}{6}\right)^{x-1} \cdot \frac{1}{6} \quad - \frac{1}{6}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 \cdots \left(\frac{5}{6}\right)^4\right]$$

$$= 1 - \frac{1}{6} \frac{\left[1 - (5/6)^5\right]}{1 - 5/6}$$

$$= (5/6)^5 = 0.4019$$

(B)

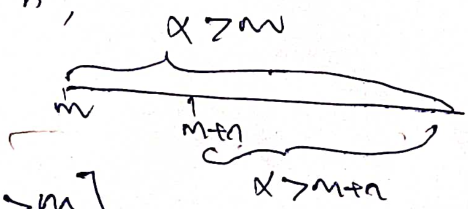
$$a + ar + ar^2 \ldots ar^{n-1} = a\left[\frac{1-r^n}{1-r}\right].$$

$$a\left[1 + r + r^2 \ldots r^{n-1}\right]$$

Establish the memoryless property of geometric distribution

Soln: If X has a geometric distribution, then for any two positive integers 'm' & 'n',

$$P[X > m+n \mid X > m] = P[X > n]$$



Proof:

$$P[\underbrace{X > m+n}_{A} \mid \underbrace{X > m}_{B}] = \frac{P[X > m+n \cap X > m]}{P[X > m]}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P[X > m+n]}{P[X > m]}$$

Taking $P[X = x] = q^{x-1}p, \; x = 1, 2, 3 \ldots$

$$P[X > k] = \sum_{k+1}^{\infty} q^{x-1} p$$

$$= q^k p + q^{k+1} p + q^{k+2} p + \ldots$$

$$= q^k p[1 + q + q^2 + \ldots]$$

$$= q^k p[1-q]^{-1}$$

$$= \frac{q^k p}{1-q}$$

$$= \frac{q^k p}{p} = q^k$$

$\therefore P[X > m+n] = q^{m+n}$ & $P[X > m] = q^m$

$$\therefore P[X > m+n / X > m] = \frac{q^{m+n}}{q^m} = q^n$$

$\therefore \underline{P[X > n] = q^n}$

Hence $P[X > m+n / X > m] = P[X > n]$

Uniform distribution (or) Rectangular distribution.

A random variable X is said to follow uniform (or) rectangular distribution over an interval (a,b) if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$$

Here 'a' & 'b' (b>a) are the parameters of the distribution.

Distribution function:

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{a}^{x} \frac{1}{b-a}\,dx$$

$$= \frac{x-a}{b-a}$$

$$\text{Hence } F(x) = \begin{cases} 0 & \text{for } x<a \\ \frac{x-a}{b-a} & \text{for } a<x<b \\ 1 & \text{for } x>b. \end{cases}$$

Find the MGF of uniform distribution:

Solu:

pdf of uniform distribution is

$$f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$$

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{a}^{b} e^{tx} \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{e^{bt}-e^{at}}{(b-a)t}$$

Mean & variance of the uniform distribution:

The pdf of uniform distribution is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$$

Mean $E(X) = \int x f(x) dx$

$$= \frac{1}{b-a} \int_a^b x dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E[X^2] = \frac{1}{b-a} \int_a^b x^2 dx$$

$$= \frac{1}{b-a} \left[\frac{x^3}{3} \right]_a^b$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{1}{3}(b^2+ab+a^2) - \frac{1}{4}[a^2+2ab+b^2]$$

$$= \frac{1}{12}[b^2-2ab+a^2]$$

$$= \frac{(b-a)^2}{12}$$

$\therefore$ Mean $= \frac{b+a}{2}$ & variance $= \frac{(b-a)^2}{12}$

Problems:

1. If x is uniformly distributed with mean 1 & variance $\frac{4}{3}$, find $p[x<0]$.

Soln:

W.K.T Mean & variance of uniform distribution over (a,b) $E(x) = \frac{b+a}{2}$ &

$$V(x) = \frac{(b-a)^2}{12}$$

Given $\frac{b+a}{2} = 1 \Rightarrow a+b=2$ ——— (1)

& $\frac{(b-a)^2}{12} = \frac{4}{3} \Rightarrow (b-a)^2 = 16$

$\Rightarrow b-a=4$ ——— (2)

Solving (1) & (2) we get

$b=3,\ a=-1$

& pdf of $f(x) = \frac{1}{b-a} = \frac{1}{4}$

(i.e) $f(x) = \begin{cases} \frac{1}{4}, & -1<x<3 \\ 0, & \text{otherwise} \end{cases}$

$\therefore p[x<0] = \int_{-1}^{0} \frac{1}{4}\,dx = \frac{1}{4}[x]_{-1}^{0} = \frac{1}{4}$ //

2. A random variable x has an uniform distribution over the interval $(-3,3)$. Compute (i) $p[x=2]$ (ii) $p[x<2]$ (iii) $p[|x|<2]$ (iv) $p[|x-2|<2]$ (v) Find k s.t $p[x>k]=1/3$

Soln:

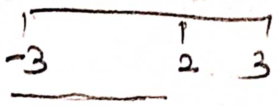
pdf of uniform distribution is $f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$

Given $b=3$ & $a=-3$

$\therefore$ pdf of x is $f(x) = \begin{cases} \frac{1}{6}, & -3<x<3 \\ 0, & \text{otherwise} \end{cases}$

(i) $P[X=2]=0$ Since the probability at a point in case of a continuous RV X is zero.

(ii) $P[X<2] = \int_{-3}^{2} \frac{1}{6}\, dx = 5/6$



(iii) $P[|x|<2] = P[-2<x<2]$

$= \frac{2+2}{6} = 2/3$

(iv) $P[|x-2|<2] = P[-2<(x-2)<2]$

$= P[0<x<4]$

$= P[0<x<3]$

$= \int_0^3 \frac{dx}{6} = 1/2$

(v) $P[X>k] = \int_k^3 \frac{1}{6}\, dx = 1/3$

$\frac{3-k}{6} = 1/3$

$\therefore 3-k = 2$

$\Rightarrow k=1$

3. Buses arrive at a specified bus stop at 15 minutes intervals starting at 7 am that is 7am, 7.15 am, 7.30am, ... If a passenger arrived at the bus stop at a random time which is uniformly distributed b/w 7 and 7.30 am find the probability that he waits ~~or~~ (a) less than 5 minutes (b) at least 12 minutes for a bus.

Soln:

Let x denotes the time that a passenger arrives b/w 7 and 7.30 am

Then $X \sim U(0, 30)$

Then $f(x) = \frac{1}{b-a} = \frac{1}{30-0} = 1/30$

(a) passenger waits less than 5 minutes

he arrives b/w 7.10 - 7.15 (or) 7.25 - 7.30

p(waiting time less than 5 minutes)

$= p[10 \le x \le 15] + p[25 \le x \le 30]$

$= \int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx$

$= \frac{1}{30}[x]_{10}^{15} + \frac{1}{30}[x]_{25}^{30} = 1/3$ //

(b) Passenger waits atleast 12 minutes

(ie) he arrives b/w 7 - 7.03 (or) 7.15 - 7.18

$\therefore$ p(waiting time atleast 12 minutes)

$= p[0 \le x \le 3) + p(15 \le x \le 18)$

$= \int_{0}^{3} \frac{1}{30} dx + \int_{15}^{18} \frac{1}{30} dx$

$= \frac{1}{30}[x]_{0}^{3} + \frac{1}{30}[x]_{15}^{18} = 1/5$ //

Exponential distribution:

A continuous R.V X is said to follow an exponential distribution if its pdf is given by $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$.

Distribution of exponential distribution:

By def. $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$$= \lambda \int_0^x e^{-\lambda x}\,dx$$

$$\therefore F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

~~Ex~~ Find MGF of Exponential distribution hence find its mean & variance.

pdf of exponential distribution is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$M_X(t) = \int e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x}\,dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^\infty \qquad e^{-\infty} = 0$$

$$= \lambda \left[0 + \frac{1}{\lambda - t} \right]$$

$$\therefore M_X(t) = \left(\frac{\lambda}{\lambda - t} \right)$$

$$M_x'(t) = \frac{(\lambda-t)0 - \lambda(-1)}{(\lambda-t)^2}$$

$$= \lambda/(\lambda-t)^2$$

$$M_x''(t) = \frac{(\lambda-t)^2 0 - \lambda 2(\lambda-t)(-1)}{(\lambda-t)^4}$$

$$= \frac{\cancel{+} 2\lambda(\lambda-t)}{(\lambda-t)^4}$$

$$= \frac{2\lambda}{(\lambda-t)^3}$$

$\mu_1' =$ Mean $M_x'(0) = \frac{\lambda}{\lambda^2} = \cancel{\lambda} \frac{1}{\lambda}$.

$$\mu_2' = M_x''(0) = \frac{2\lambda}{\lambda^3} = 2/\lambda^2$$

Hence mean $\boxed{\mu_1' = 1/\lambda}$

Variance $\mu_2 = \mu_2' - \mu_1'^2$

$$= 2/\lambda^2 - 1/\lambda^2$$

$$= 1/\lambda^2$$

$\therefore \boxed{\text{Var} = 1/\lambda^2}$

(8)

State & prove Memoryless Property of exponential distribution.

State:

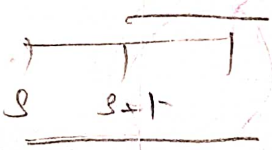
If X is exponentially distributed R.V, then
$p[X > s+t \,/\, X > s] = p[X > t]$, for any $s, t > 0$.

Proof:

pdf of exponential distribution is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}, \quad \lambda > 0.$$

$$P[X > k) = \int_k^\infty \lambda e^{-\lambda x} dx$$



$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_k^\infty$$

$$= e^{-\lambda k} \quad \text{——— (1)}$$

$$p[X > s+t \,|\, X > s) = \frac{p[X > s+t \;\&\; X > s]}{p[X > s]}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t} = p[X > t)$$

Hence $p[X > s+t \,|\, X > s) = p[X > t)$

The converse of this result is also true.

(ie) If $p[X > s+t \,/\, X > s) = p(X > t)$ then X follows an exponential distribution.

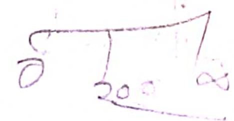
1. The mileage which car owner get with a certain kind of radial tire is a Rv having an exponential distribution with mean 40,000 km. Find the probabilities that one of these tires will last (i) atleast 20,000 km or (ii) atmost 30,000 km.

<u>Soln:</u>

Let x denote the mileage obtained with the tire & mean $= 1/\lambda$.

$\therefore$ Mean $\frac{1}{\lambda} = 40,000$ km (given)

$\therefore \lambda = \frac{1}{40,000}$



The density fun. $f(x)$ is given by $f(x) = \lambda e^{-\lambda x}, x \geq 0$

$\therefore f(x) = \frac{1}{40,000} e^{-\frac{1}{40,000}x}$

(i) P[one of the tires will last atleast 20,000 km]

$$P[x \geq 20,000] = \int_{20,000}^{\infty} \frac{1}{40,000} e^{-\frac{1}{40,000}x} dx$$

$$= -\frac{1}{40,000}\left[40,000\, e^{-x/40,000}\right]_{20,000}^{\infty}$$

$$= e^{-0.5} = 0.6065$$

(ii) P[one of those tire will last atmost 30,000 km]

$$\therefore P[x \leq 30,000] = \int_{0}^{30,000} \frac{1}{40,000} e^{-x/40,000} dx$$

$$= -\frac{1}{40,000}\left[40,000\, e^{-x/40,000}\right]_{0}^{30,000}$$

$$= 1 - e^{-0.75}$$

$$= 0.5270$$

2\. The time in hours required to repair a machine is exponentially distributed with parameter $\lambda = 1/2$

(i) what is the probability that the repair time exceeds 2h,

(ii) What is the conditional probability that a repair takes atleast 10h given that its duration exceeds 9h?

Soln:

Given $\lambda = 1/2$

Let 'x' denote the time to repair the machine

The density function of x is given by,

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$= \frac{1}{2} e^{-\frac{1}{2}x}, \quad x \geq 0$$

(i) p(the repair time exceeds 2h) = $p(X > 2)$

$$= \int_2^\infty \frac{1}{2} e^{-\frac{1}{2}x} dx$$

$$= -\frac{1}{2} \cdot \left[2e^{-\frac{1}{2}x}\right]_2^\infty$$

$$= e^{-1} = 0.3679.$$

(ii) The conditional probability that a repair takes atleast 10h given that its duration exceeds 9h is given by.

$$p[X > 10 / X > 9] = p[X > 9+1 / X > 9] = p[X > 1] \quad \text{by M.L.P.}$$

$$p(A/B) = \frac{p(A \cap B)}{p(B)}$$

$$= \int_1^\infty \frac{1}{2} e^{-\frac{1}{2}x} dx$$

$$= e^{-1/2}$$

$$= 0.6065.$$

Gamma distribution

A continuous R.V X is said to follow a gamma distribution with parameter n, if its pdf is given by $f(x) = \begin{cases} \frac{e^{-x} x^{n-1}}{\Gamma n}, & n>0,\ 0<x<\infty \\ 0, & \text{otherwise} \end{cases}$.

Generalised Gamma distribution [Erlang distribution]

A continuous R.V X having the following density function is said to follow generalised gamma distribution with parameters α & n

$$f(x) = \begin{cases} \frac{\alpha^n}{\Gamma n} e^{-\alpha x} x^{n-1}; & \text{for } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the MGF of Gamma distribution hence find its mean & variance.

pdf of Gamma distribution is

$$f(x) = \begin{cases} \frac{e^{-x} x^{n-1}}{\Gamma n}, & n>0,\ 0<x<\infty \\ 0, & \text{otherwise} \end{cases}$$

$$M_X(t) = E[e^{tX}] = \int e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \frac{e^{-x} x^{n-1}}{\Gamma n}\,dx \qquad \left[\because \int_0^\infty e^{-ax} x^{n-1}\,dx = \frac{\Gamma n}{a^n}\right]$$

$$= \frac{1}{\Gamma n} \int_0^\infty e^{-(1-t)x} x^{n-1}\,dx$$

$$= \frac{1}{\cancel{\Gamma n}} \frac{\cancel{\Gamma n}}{(1-t)^n}$$

$$\therefore M_X(t) = \frac{1}{(1-t)^n}$$

$$\text{(i.e.) } M_X(t) = (1-t)^{-n}$$

$$M_x'(t) = -n(1-t)^{-n-1}(-1)$$
$$= n(1-t)^{-n-1}$$
$$M_x''(t) = +n(-n-1)(1-t)^{-n-2}(-1) \quad (2)$$
$$= n(n+1)(1-t)^{-n-2}$$

$$\mu_1' = M_x'(0) = n \text{ ———— } (1)$$

$$\mu_2' = M_x''(0) = n(n+1) \text{ ———— } (2)$$

$$\boxed{\text{Mean } \mu_1' = n}$$

& $\text{Variance} = \mu_2 = \mu_2' - \mu_1'^2$
$$= n(n+1) - n^2$$
$$= \cancel{n^2} + n - \cancel{n^2}$$

$$\boxed{\text{Variance} = n}$$

1. In a certain city, the daily consumption of electric power in millions of kilowatt hour can be treated as a R.V having an Erlang distribution with parameter $(1/2, 3)$. If the power plant of this city has a daily capacity of 12 millions kilowatt hours, what is the probability that this power supply will be inadequate on any given day.

Soln: Let X represent the daily consumption of electric power in millions of kilowatt hour.

The pdf of Gamma distribution is (3)

$$f(x) = \begin{cases} \alpha^n / \Gamma n \cdot e^{-\alpha x} \cdot x^{n-1}, & n>0,\ x\geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Given $\alpha = 1/2$ & $n=3$

$$\therefore f(x) = \frac{(\frac{1}{2})^3 e^{-\frac{1}{2}x} x^{3-1}}{\Gamma 3}, \quad x\geq 0$$

$$= \frac{1}{16} e^{-\frac{1}{2}x} x^2, \quad x\geq 0 \qquad [\because \Gamma 3 = 2! = 2]$$

P(the power supply is inadequate)

$$= P[X > 12] = \int_{12}^{\infty} f(x)\,dx$$

$$= \frac{1}{16} \int_{12}^{\infty} e^{-x/2} x^2\,dx$$

$$= \frac{1}{16}\left\{ x^2 \left(\frac{e^{-x/2}}{-1/2}\right) - 2x\left[\frac{e^{-x/2}}{1/4}\right) + 2\left[\frac{e^{-x/2}}{-1/8}\right]\right\}_{12}^{\infty}$$

$$= \frac{1}{16}\left[-2x^2 e^{-x/2} - 8x e^{-x/2} - 16 e^{-x/2}\right]_{12}^{\infty}$$

$$= \frac{1}{16}\left\{(-0-0-0) - [-2\times 12^2 e^{-6} - 8\times 12 e^{-6} - 16 e^{-6}]\right\}$$

$$= \frac{1}{16}[288 + 96 + 16]\, e^{-6} = 25 e^{-6}$$

$$= 0.0625$$

2. The daily consumption of milk in a city in excess of 20,000 L is approximately distributed as a Gamma Variate with parameter $\alpha = 2$ & $\lambda = \frac{1}{10,000}$. The city has a daily stock of 30,000 litres. What is the probability that the stock is insufficient on a particular day?

Soln:

If the RY Y denotes the daily consumption of milk in a city then R.V $Y = X + 20{,}000$

(ie) $X = Y - 20{,}000$, then X follows a Gamma distribution with parameter $\alpha = 2$ & $\lambda = \frac{1}{10{,}000}$ (4)

by our notation (ie) $n = 2$ & $\alpha = \frac{1}{10{,}000}$

pdf of Gamma dist. is

$$f(x) = \frac{\alpha^n}{\Gamma n}, e^{-\alpha x}, x^{n-1}, \quad 0 < x < \infty .$$

$$\therefore \quad f(x) = \left(\frac{1}{10{,}000}\right)^2 \cdot \frac{1}{\Gamma 2}\, e^{-x/10{,}000}, x^{2-1}$$

$$= \frac{1}{(10{,}000)^2} \cdot \frac{1}{1} \cdot e^{-x/10{,}000} x \quad [\Gamma 2 = 1! = 1]$$

Since the daily stock of the city is 30,000 L

The required probability that the ~~daily~~ stock is insufficient on a particular day is given by

$$P[Y > 30{,}000] = P[X + 20{,}000 > 30{,}000]$$

$$= P[X > 10{,}000]$$

$$= \int_{10{,}000}^{\infty} \frac{x\, e^{-x/10{,}000}}{(10{,}000)^2 \times 1}\, dx$$

$$= \frac{1}{(10{,}000)^2 \times 1}\left[x \frac{e^{-x/10{,}000}}{-1/10{,}000} - \frac{e^{-x/10{,}000}}{(1/10{,}000)^2} \right]_{10{,}000}^{\infty}$$

$$= \frac{1}{(10{,}000)^2}\left[(-0-0) - [-10{,}000 \times 10{,}000 e^{-1} - (10{,}000)^2 e^{-1}]\right]$$

$$= \frac{1}{(10{,}000)^2}\left[(10{,}000)^2 e^{-1} + (10{,}000)^2 e^{-1}\right]$$

$$= \frac{1}{(10{,}000)^2}\left[2(10{,}000)^2 e^{-1}\right]$$

$$= 2/e$$

Weibull distribution.

A continuous R.v X is said to follow a weibull distribution with parameter $\alpha, \beta > 0$ if it has the pdf given by $f(x) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}, & x>0 \\ 0, & \text{otherwise} \end{cases}$

Note:

When $\beta=1$, then $f(x) = \begin{cases} \alpha e^{-\alpha x}, & x>0 \\ 0, & \text{otherwise} \end{cases}$ which is the pdf of the exponential distribution.

[ies when $\beta=1$, the weibull distribution reduces to the exponential distribution with parameter α].

Mean & variance of weibull distribution:

$$\text{Mean } E(X) = \int x f(x)\,dx$$

$$= \int_0^\infty x \cdot \alpha\beta x^{\beta-1} \cdot e^{-\alpha x^\beta}\,dx$$

$$\left[\text{Put } t = \alpha x^\beta \quad dt = \alpha\beta x^{\beta-1}\,dx \quad \Rightarrow x = (t/\alpha)^{1/\beta}\right]$$

$$= \int_0^\infty \left(\frac{t}{\alpha}\right)^{1/\beta} e^{-t}\,dt$$

$$= \alpha^{-1/\beta} \int_0^\infty e^{-t}\, t^{1/\beta + 1 - 1}\,dt$$

$$E(X) = \text{Mean} = \alpha^{-1/\beta}\,\Gamma(1 + 1/\beta)$$

$$E(X^2) = \int_0^\infty x^2 \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}\,dx$$

$$\left[\text{Put } t = \alpha x^\beta \quad dt = \alpha\beta x^{\beta-1}\,dx \quad \Rightarrow x = (t/\alpha)^{1/\beta} \quad x=0 \Rightarrow t=0, \; x=\infty \Rightarrow t=\infty\right]$$

$$= \int_0^\infty \left(\frac{t}{\alpha}\right)^{2/\beta} e^{-t} dt$$

$$= \alpha^{-2/\beta} \int_0^\infty e^{-t} t^{2/\beta+1-1} dt$$

$$E(x^2) = \alpha^{-2/\beta} \Gamma(1+2/\beta)$$

$$\therefore Var(x) = E(x^2) - [E(x)]^2$$

$$= \alpha^{-2/\beta} \Gamma(1+2/\beta) - \alpha^{-2/\beta} \left[\Gamma(1+1/\beta)\right]^2$$

Problem:

1. Suppose that lifetime of certain kind of an emergency backup battery (in hrs) is a RV x having the weibull distribution $\alpha = 0.1$ & $\beta = 0.5$ find

(a) the mean lifetime of three batteries.

(b) the probability that such a battery will last more than 300 hrs.

(c) the probability that such a battery will not last 100 hrs.

Soln:

Let x denotes the lifetime of an emergency backup battery. Then its pdf is

$$f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}, x > 0$$

Given $\alpha = 0.1$ & $\beta = 0.5$

$$\Rightarrow f(x) = (0.1)(0.5) x^{0.5-1} e^{-0.1 x^{0.5}}, x > 0$$

$$= (0.05) x^{-0.5} e^{-0.1 x^{0.5}}, x > 0$$

a) $E(x) = \alpha^{-1/\beta} \Gamma(1+1/\beta)$

$$= (0.1)^{-1/0.5} \Gamma\left(1+\frac{1}{0.5}\right) = (0.1)^{-2} \Gamma 3$$

$$= \frac{1}{0.01} \times (2) = 200 \text{ hrs}.$$

(b) $P(X>300) = \int_{300}^{\infty} (0.05)\, x^{-0.5} e^{-0.1 x^{0.5}} dx$

$\left[\text{put } t = 0.1 x^{0.5};\ dt = (0.1)(0.5) x^{-0.5} dx \right.$

when $x=\infty$; $t=\infty$; when $x=300 \Rightarrow t=(0.1)(300)^{1/2}=\sqrt{3}$ $]$

$= \int_{\sqrt{3}}^{\infty} e^{-t} dt$

$= \left[-e^{-t}\right]_{\sqrt{3}}^{\infty} = e^{-\sqrt{3}} = 0.177$

(c) $P(X<100) = \int_{0}^{100} (0.05)\, x^{-0.5} e^{-0.1 x^{0.5}} dx$

$\left[\text{put } t = (0.1) x^{0.5} \Rightarrow x=0 \Rightarrow t=0 \right.$
$x=100 \Rightarrow t=(0.1)(100)^{0.5}=1$ $]$

$dt = (0.1)(0.5) x^{-0.5} dx$ $]$

$= \int_{0}^{1} e^{-t} dt$

$= \left[-e^{-t}\right]_{0}^{1}$

$= 1-e^{-1} = 0.6321$

2. Each of the 6 tubes of a radio set has a life length (in yrs). which may be considered as a RV that follows a weibull distribution with parameter $\alpha=25$ & $\beta=2$. If there tubes function independently of one another, what is the probability that no tube will have to be replaced during the first 2 months of service.

Soln:-

Let X represents the life length of each tube

Then its pdf is $f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}, x>0$

Given $\alpha=25$ & $\beta=2$

$\therefore f(x) = 50\, x e^{-25x^2}, x>0$

p(a tube is not to be replaced during the 1st 2 months) $= p[X > 2/12)$

$= p[x > 1/6]$

$= \int_{1/6}^{\infty} 50x\, e^{-25x^2} dx$

[put $25x^2 = t$ $\quad x = \infty \Rightarrow t = \infty$

$50x\,dx = dt$ $\quad x = 1/6 \Rightarrow t = 25/36$]

$= \int_{25/36}^{\infty} e^{-t} dt = \left[-e^{-t}\right]_{25/36}^{\infty} = e^{-25/36} = 0.4993$ //

$\therefore$ p(all the 6 tubes are not to be replaced during the 1st 2 months $= \left(e^{-25/36}\right)^6$ [by indepen]

$= e^{-25/6} = 0.0155$ //

3. If the life x (in yrs) of a certain type of car has a weibull distribution with the parameter $\beta = 2$, find the value of the parameter α, given the probability that the life of the car exceeds 5 years is $e^{-0.25}$ for these values of α & β, find the mean & variance of x.

Solu: Given $\beta = 2$. The density function of x is given by $f(x) = 2\alpha x e^{-\alpha x^2}$, $x > 0$

$\therefore p[X > 5] = \int_{5}^{\infty} 2\alpha x\, e^{-\alpha x^2} dx$

$\left[\text{Put } t = \alpha x^2 \Rightarrow x = (t/\alpha)^{1/2} \right.$

$dt = 2\alpha x\,dx \qquad x = 5 \Rightarrow t = 25\alpha$

$\left. \qquad\qquad\qquad x = \infty \Rightarrow t = \infty \right]$

$$= \int_{25\alpha}^{\infty} e^{-t}\,dt$$

$$= \left[\frac{e^{-t}}{-1} \right]_{25\alpha}^{\infty} = e^{-25\alpha} \quad \text{---} (1)$$

Given that $P(X>5) = e^{-0.25}$ —— (2)

from (1) & (2) $\quad e^{-25\alpha} = e^{-0.25}$

$$\Rightarrow \alpha = \frac{1}{100}$$

Mean of weibull distribution is

$$E(X) = \alpha^{-1/\beta}\,\Gamma\left(1/\beta + 1\right)$$

$$= \left(\frac{1}{100}\right)^{-1/2} \Gamma\left(1/2 + 1\right)$$

$$= 10\,\Gamma 3/2 = 10\,\tfrac{1}{2}\,\Gamma 1/2 \qquad [\because \Gamma 1/2 = \sqrt{\pi}]$$

$$= 10 \cdot 1/2\, \Gamma 1/2$$

$$= 5\sqrt{\pi}$$

$$\text{Var}(X) = e^{-2/\beta}\left[\Gamma(2/\beta + 1) - (\Gamma(1/\beta + 1))^2 \right]$$

$$= \left(\frac{1}{100}\right)^{-1} \left[\Gamma 2 - (\Gamma(1/2 + 1))^2 \right]$$

$$= 100 \left[1 - (\Gamma 3/2)^2 \right]$$

$$= 100 \left[1 - (1/2\, \Gamma 1/2)^2 \right]$$

$$= 100 \left(1 - (1/2 \sqrt{\pi})^2 \right]$$

$$= 100 \left[1 - (\sqrt{\pi}/2)^2 \right]$$

$$= 100 (1 - \pi/4)$$