Newton's interpolation formula for unequal intervals

Let $y = f(x)$ take values $f(x_0), f(x_1), \ldots, f(x_n)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$

By definition

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\therefore f(x) = f(x_0) + (x - x_0) f(x, x_0) \quad \text{--- (1)}$$

Illy $$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

$$\therefore f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1)$$

Using this value of $f(x, x_0)$ in (1), we have

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x, x_0, x_1)$$

Again $$f(x, x_0, x_1, x_2) = \frac{f(x, x_0, x_1) - f(x_0, x_1, x_2)}{x - x_2}$$

$$\therefore f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1, x_2)$$

Using this value in (2), we get

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x, x_0, x_1, x_2) \quad \text{--- (3)}$$

Continuing in this manner, we get

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \ldots + (x - x_0)(x - x_1)(x - x_2) \ldots (x - x_{n-1}) f(x_0, x_1, x_2, \ldots x_n) + (x - x_0)(x - x_1)(x - x_2) \ldots (x - x_n) f(x, x_0, x_1, \ldots x_n)$$

(15)

If $f(x)$ is a polynomial of degree n, then $f(x, x_0, x_1, \ldots x_n) = 0$ { $(n+1)^{th}$ diff }

∴ (A') becomes.

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) \ldots (x-x_0)(x-x_1) \ldots (x-x_{n-1}) f(x_0, x_1, \ldots) \quad (B)$$

Equ (B) is called Newton's divided difference interpolation formula for unequal intervals.

Problem:

Using Newton's divided difference formula, find the values of $f(2)$, $f(8)$ & $f(15)$ given the following table.

x:	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

Soln. Divided difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
4	48			
		$\frac{100-48}{5-4} = 52$		
5	100		$\frac{97-52}{7-4} = 15$	
		97		1
7	294		21	
		202		1
10	900		27	
		310		1
11	1210		33	
		407		
13	2028			

By Newton's divided diff. interpolation formula,

$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + \ldots$ ①

Given $x_0 = 4$, $x_1 = 5$, $x_2 = 7$, $x_3 = 10$, $x_4 = 11$ & $x_5 = 13$

& $f(x_0) = 48$, $f(x_0, x_1) = 52$, $f(x_0, x_1, x_2) = 15$, $f(x_0, x_1, x_2, x_3) = 1$

Hence using these values in ①, we have

$f(x) = 48 + (x-4)52 + (x-4)(x-5)15 + (x-4)(x-5)(x-7)$

$f(2) = 48 - 104 + 90 - 30 = 4$

$f(8) = 48 + (4)52 + (4)(3)15 + 4(3)(1)(1) = 448$

$f(15) = 48 + 11\times 52 + 11\times 10\times 15 + 11\times 10\times 8 = 3150$

2) From the following table find $f(x)$ and hence $f(6)$ using Newton's interpolation formula

x :	1	2	7	8
$f(x)$:	1	5	5	4

Soln: Evidently, intervals are not equal. We form the divided difference table below

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1			
		4		
2	5		$-2/3$	
		0		$\frac{1}{14}$
7	5		$-1/6$	
		-1		
8	4			

By Newton's divided difference formula,

$f(x) = f(x_0) + (x-x_0) f(x_1, x_0) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3)$

$= 1 + (x-1)4 + (x-1)(x-2)\left(-\frac{2}{3}\right) + (x-1)(x-2)(x-7)\left(\frac{1}{14}\right)$

$= \frac{1}{42}\left(3x^3 - 58x^2 + 311x - 224\right)$

$f(6) = \frac{1}{42}\left[3\times 216 - 36\times 58 + 1926 - 224\right]$

$= 6.23809524$.