Cubic Spline interpolation

We define a Cubic Spline, $S(x)$ as follows.

(i) $S(x)$ is a polynomial of degree one for $x < x_0$ and $x > x_n$.

(ii) $S(x)$ is almost a Cubic polynomial in each interval (x_{i-1}, x_i), $i = 1, 2, \dots n$

(iii) $S(x)$, $S'(x)$ and $S''(x)$ are continuous at each point (x_i, y_i), $i = 0, 1, 2 \dots n$ and

(iv) $S(x_i) = y_i$, $i = 0, 1, 2 \dots n$

Method 1.

For Convenience, we assume equal interval (i.e) $x_i - x_{i-1} = h$, $i = 1, 2, 3 \dots n$. Since there are n equal intervals, we have to find n cubic polynomials totally. Hence, if the number of intervals is large, it is not easy to find all these polynomials - cubic splines.

Since $S(x)$ is a Cubic polynomial, $S''(x)$ is linear in each interval.

In the interval (x_{i-1}, x_i),

Let us assume

$$S''(x) = \frac{1}{h}\left[(x_i - x) S''(x_{i-1}) + (x - x_{i-1}) S''(x_i)\right] \quad (1)$$

We can easily check that this equation is valid when we put $x = x_{i-1}$ & $x = x_i$.

Integrating twice

$$S(x) = \frac{1}{h}\left[\frac{(x_i-x)^3}{3!} S''(x_{i-1}) + \frac{(x-x_{i-1})^3}{3!} S''(x_i)\right] + a_i(x_i-x) + b_i(x-x_{i-1}) \quad (2)$$

Where a_i, b_i are constants to be found out by using the conditions,

$S(x_i) = y_i$ (given), $i = 0, 1, 2 \dots n$

put $x = x_{i-1}$ in (2), we get

$$y_{i-1} = \frac{1}{h}\left[\frac{h^3}{3!} S''(x_{i-1})\right] + h a_i$$

$$\therefore a_i = \frac{1}{h}\left\{y_{i-1} - \frac{h^2}{3!} S''(x_{i-1})\right\}$$

put $x = x_i$ in (2), we get

$$b_i = \frac{1}{h}\left[y_i - \frac{h^2}{3!} S''(x_i)\right]$$

Hence the equation (2) reduces to

$$S(x) = \frac{1}{h}\left\{\frac{(x_i-x)^3}{3!} S''(x_{i-1}) + \frac{(x-x_{i-1})^3}{3!} S''(x_i)\right\} + \frac{1}{h}(x_i-x)\left\{y_{i-1} - \frac{h^2}{3!} S''(x_{i-1})\right\} + \frac{1}{h}(x-x_{i-1})\left\{y_i - \frac{h^2}{3!} S''(x_i)\right\} \quad (3)$$

Writing $S''(x_i) = M_i$, the above equation becomes

$$S(x) = \frac{1}{6h}\left[(x_i-x)^3 M_{i-1} + (x-x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i-x)\left[y_{i-1} - \frac{h^2}{6} M_{i-1}\right] + \frac{1}{h}(x-x_{i-1})\left[y_i - \frac{h^2}{6} M_i\right] \quad ($$

The quantities M_i which are the spline second derivatives are not yet known.

Now we will impose the continuity of $s'(x)$

from ④ $s'(x) = \frac{1}{6h}\left[3(x_i - x)^2(-M_{i-1}) + 3(x - x_{i-1})^2 M_i\right]$

$$+ \frac{1}{h}\left[-y_{i-1} + \frac{h^2}{6}M_{i-1}\right] + \frac{1}{h}\left[y_i - \frac{h^2}{6}M_i\right]$$

$\therefore s'(x_{i-}) = \frac{h}{3}M_i + \frac{h}{6}M_{i-1} + \frac{1}{h}(y_i - y_{i-1})$ ⑤

l^{ly} $s'(x_{i+}) = -\frac{h}{3}M_i - \frac{h}{6}M_{i+1} + \frac{1}{h}(y_{i+1} - y_i)$ ⑥

Equating ⑤ & ⑥, we get

$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right]$

for $i = 1, 2, 3 \dots (n-1)$ ⑦

Further, in view of the first condition, that $s(x)$ is linear for $x < x_0$ & $x > x_n$.

We have $s''(x) = 0$ at $x = x_0$ & $x = x_n$.

Hence $M_0 = 0$, $M_n = 0$ ⑧

Equ. ⑦ & ⑧ give $(n+1)$ equ. in $(n+1)$ unknown $M_0, M_1, \dots M_n$. Hence we can solve for $M_0, M_1, M_2, \dots M$.

Substituting in ④, we get the cubic spline in each interval.