3. Find the cubic spline approximation for the function given below

x	0	1	2	3
y	1	2	33	244

Assume $M(0) = M(3) = 0$. Also find $y(2.5)$.

Soln: Here $h = 1$, & $n = 3$.

We have $M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}]$ —— (1) for $i = 1, 2$

$\therefore M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2]$

$M_1 + 4M_2 + M_3 = 6[y_1 - 2y_2 + y_3]$

These reduces to, [taking $M_0 = 0$, $M_3 = 0$]

$4M_1 + M_2 = 6(1 - 4 + 33) = 180$

$M_1 + 4M_2 = 6(2 - 66 + 244) = 1080$

Solving $M_1 = -24$, $M_2 = 276$.

W.k.T

$$s(x) = \frac{1}{6h}\{(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\} + \frac{1}{h}(x_i - x)\{y_{i-1} - \frac{h^2}{6}M_{i-1}\}$$
$$+ \frac{1}{h}(x - x_{i-1})\{y_i - \frac{h^2}{6}M_i\}$$
for $i = 1, 2, 3 \cdots n$ —— (2)

Put $i = 1$ in eqn. (2)

$$s(x) = \frac{1}{6h}\{(x_1 - x)^3 M_0 + (x - x_0)^3 M_1\} + \frac{1}{h}(x_1 - x)(y_0 - \frac{h^2}{6}M_0)$$
$$+ \frac{1}{h}(x - x_0)(y_1 - \frac{h^2}{6}M_1)$$
$$= \frac{1}{6}[0 + (x-0)^3(-24)] + (1-x)[1-0] + (x-0)[2 + \frac{24}{6}]$$
$$= -4x^3 + 5x + 1$$ —— (3)

In [1,2], [put i = 2 in (3)]

$$S(x) = \frac{1}{6}\left[(2-x)^3(-24) + (x-1)^3(276)\right] + (2-x)\left[2 + \frac{24}{6}\right] + (x-1)\left[33 - \frac{276}{6}\right]$$

$$= 50x^3 - 162x^2 + 167x - 53 \quad \text{—— II}$$

In [2,3] [put i = 3 in (3)]

$$S(x) = \frac{1}{6}\left\{(3-x)^3 M_2\right\} + (3-x)\left(33 - \frac{M_2}{6}\right) + (x-2)(244 - 0)$$

$$= \frac{1}{6}\left\{(3-x)^3(276)\right\} + (3-x)(33-46) + (x-2)(244)$$

$$= 46(3-x)^3 - 13(3-x) + 244(x-2)$$

$$= -46x^3 + 414x^2 - 985x + 715 \quad \text{—— III}$$

Eqn. I, II & III give the cubic spline in each sub interval.

$y(2.5) = 121.25$ //

(4) Test whether the following fun. are cubic splines (or) not

$P_1(x) = x^2 - x + 1, \quad 1 \le x \le 2$

$P_2(x) = 3x - 3, \quad 2 \le x \le 3$

Each poly. is at most of degree three in each sub-interval.

$P_1(2) = 3 = P_2(2)$

$P_1'(2) = 3 = P_2'(2)$

$P_1''(2) = 2, \quad P_2''(2) = 0$

$\therefore$ Not a cubic spline since $S''(x)$ is not continuous at $x = 2$.