UNIT – III

Numerical differentiation & Integration

Numerical differentiation

Numerical differentiation is the process of computing the value of $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ for some particular value of x from the given data (x_i, y_i), $i = 1, 2, \ldots n$ where $y = f(x)$ is not known explicitly.

Newton's forward difference formula to get the derivative.

Newton's forward difference interpolation formula is

$y(x) = y(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots$

where $y(x)$ is a poly. of degree in 'n' in x

& $u = \frac{x - x_0}{h}$.

Diff. $y(x)$ w.r.to x.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{h} \frac{dy}{du}.$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6}\Delta^3 y_0 + \ldots\right] \quad \text{—(2)}$$

Diff (2) again w.r.to x

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{6u^2 - 18u + 11}{12}\right)\Delta^4 y_0 + \ldots\right] \quad \text{—(3)}$$

$$\text{Hence } \frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{12u - 18}{12}\Delta^4 y_0 + \ldots\right] \quad \text{—(4)}$$

Eqn. (2), (3) & (4) give the 1st, 2nd & 3rd derivative at any x

Setting $x = x_0$ (ie) $u = \frac{x - x_0}{h} = 0$ in (2), (3) & (4)

we get

$$\frac{dy}{dx}\Big|_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \dots\right]$$

$$\frac{d^2y}{dx^2}\Big|_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \dots\right]$$

Also

$$\frac{d^3y}{dx^3}\Big|_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \dots\right]$$

The above 3 equ. are the 1st, 2nd & 3rd derivative at the starting value $x = x_0$

Newton's backward diff. formula to compute the derivative

Now, consider Newton's backward diff. interpolation formula.

$$y(x) = y(x_n + vh) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \dots$$

where $v = \frac{x - x_n}{h}$ ——— (1)

Diff. (1) w.r.to x.

$$\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx} = \frac{dy}{dv}\cdot\frac{1}{h}$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2v+1}{2}\nabla^2 y_n + \frac{3v^2+6v+2}{6}\nabla^3 y_n + \frac{4v^3+18v^2+22v+6}{24}\nabla^4 y_n\right.$$ ——— (2)

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \frac{6v^2+18v+11}{12}\nabla^4 y_n + \dots\right]$$ ——— (3)

$$\therefore \frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{12v+18}{12}\nabla^4 y_n + \dots\right]$$ ——— (4)

Equ. (2), (3) & (4) give the 1st, 2nd & 3rd derivative at any general x

Setting $x = x_n$ (or) $v = 0$ in eq. (3) & (4)

We get

$$\left.\frac{dy}{dx}\right|_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right] \quad (5)$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right] \quad (6)$$

$$\left.\frac{d^3y}{dx^3}\right|_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right] \quad (7)$$

Eqn. (5), (6) & (7) give the 1st, 2nd & 3rd derivatives at $x = x_n$.

1) Find the 1st two derivatives of $(x)^{1/3}$ at $x=50$ & $x=56$, given the table below.

x :	50	51	52	53	54	55	56
$y = x^{1/3}$	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Soln. Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
50	3.6840			
		0.0244		
51	3.7084		-0.0003	
		0.0241		0
52	3.7325		-0.0003	
		0.0238		0
53	3.7563		-0.0003	
		0.0235		0
54	3.7798		-0.0003	
		0.0232		0
55	3.8030		-0.0003	
		0.0229		
56	3.8259			

Since we require $f'(x)$ at $x=50$ we use Newton forward formula & to get $f'(x)$ at $x=56$ we use Newton's backward formula.

By Newton's forward formula.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \dots\right]$$

$$= 1\left[0.0244 - \frac{1}{2}(-0.0003) + \frac{1}{3}(0)\right]$$

$$= 0.02455$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=50} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \dots\right]$$

$$= 1[-0.0003] = -0.0003$$

By Newton's backward diff formula.

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \left(\frac{dy}{dx}\right)_{v=0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \dots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=50} = \frac{1}{1}\left[0.0229 + \frac{1}{2}(-0.0003) + 0\right]$$

$$= 0.02275$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=56} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \dots\right]$$

$$= \frac{1}{1}[-0.0003]$$

$$= -0.0003$$