Numerical Integration

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called numerical integration. This process when applied to a function of a single variable, is known as quadratic.

Newton-cotes quadrature formula:

Let $I = \int_a^b f(x)\,dx$

Where $f(x)$ takes the $y_0, y_1, \ldots y_n$ for $x = x_0, x_1, x_2, \ldots x_n$.

Let us divide the interval (a,b) into n sub-intervals of width h so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h \ldots x_n = x_0 + nh = b$

Then $I = \int_{x_0}^{x_0+nh} f(x)\,dx = h \int_0^n f(x_0 + rh)\,dr$

putting $x = x_0 + rh$, $dx = h\,dr$

$$= h\int_0^n \left\{y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots\right\} dr$$

[by Newton's forward interpolation formula]

Integrating term by term, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = nh\left\{y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0\right.$$

$$+\left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11}{3}n^2 - 3n\right)\frac{\Delta^4 y_0}{4!}$$

$$+\left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50}{3}n^2 + 12n\right)\frac{\Delta^5 y_0}{5!}$$

$$\left.+\left(\frac{n^6}{7} - \frac{15n^5}{6} + 17n^4 - \frac{225}{4}n^3 + \frac{274}{3}n^2 - 60n\right)\frac{\Delta^6 y_0}{6!} + \ldots\right\} \quad (1)$$

This is known as Newton cotes quadrature formula.

Trapezoidal rule:

Put $n=1$ in (1)

$$\int_{x_0}^{x_0+h} f(x)\,dx \approx h\left\{1\cdot y_0 + \frac{1}{2}\Delta y_0\right\}$$

Since other differences do not exist if $n=1$.

$$\approx h\left\{y_0 + \frac{1}{2}(y_1 - y_0)\right\}$$

$$= \frac{h}{2}[y_0 + y_1]$$

Ily
$$\int_{x_0+h}^{x_0+2h} f(x)\,dx \approx h\left[y_1 + \frac{1}{2}\Delta y_1\right] = \frac{h}{2}(y_1 + y_2)$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1}+y_n)$$

Adding these 'n' integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{20}\left\{(y_0+y_n)+2(y_1+y_2+\cdots y_{n-1})\right\} \quad (2)$$

This is known as the trapezoidal rule.

(ie) $\frac{h}{2}$ [Sum of the 1st & last ordinates + 2 (Sum of remaining ordinates)]

Simpson's one third rule:

Setting $n=2$ in Newton's-Cotes' quadrature formula we have

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left(y_0+\Delta y_0+\frac{1}{6}\Delta^2 y_0\right)$$

$$=\frac{h}{3}(y_0+4y_1+y_2)$$

Illy
$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}(y_2+4y_3+y_4)$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2}+4y_{n-1}+y_n), \text{ n being even.}$$

Adding all these, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left\{(y_0+y_n)+4(y_1+y_3+\cdots y_{n-1}) + 2(y_2+y_4+\cdots y_{n-2})\right\} \quad (3)$$

This is known as Simpson's $\frac{1}{3}$rd rule.

Simpson's three eight rule.

Put $n=3$ in (1)

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{2}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$\approx \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Ily

$$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on.}$$

Adding all such expressions from x_0 to x_0+nh, where n is a multiple of 3, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left\{(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\dots y_{n-1}) + 2(y_3+y_6+\dots y_{n-3})\right\}$$

which is known as Simpson's $3/8$th rule.

Rule	Degree of y(x)	No. of intervals	Error	Order
Trapezoidal rule	One	any	$\lvert E\rvert < \frac{(b-a)}{12}h^2 \cdot M$	h^2
Simpson's $\frac{1}{3}$ rule	Two	even	$\lvert E\rvert < \frac{(b-a)}{180}h^4 \cdot M$	h^4
Simpson's $\frac{3}{8}$ rule	Three	Multiple of 3	$\lvert E\rvert < \frac{b-a}{80}h^4 \cdot M$	h^4.

1. Evaluate $\int_0^5 \frac{dx}{4x+5}$ by Simpson's one third rule and hence find the value of $\log_e 5$ $(n=10)$

Soln

Here $y(x) = \frac{1}{4x+5}$

$h = \frac{5-0}{10} = \frac{1}{2}$

x	0	0.5	1	1.5	2	2.5	3
y	0.2	0.1429	0.1111	0.0909	0.0769	0.0667	0.0588
				3.5	4	4.5	5
				0.0526	0.0476	0.0434	0.04

By Simpson's 1/3 rd rule

$$\int_0^5 \frac{dx}{4x+5} = \frac{h}{3}\left\{(y_0+y_n)+2(y_2+y_4+y_6+\ldots)+4(y_1+y_3+y_5+\ldots)\right\}$$

$$= \frac{1}{6}\left\{(0.2+0.04)+2(0.1111+0.0769+0.0588+0.0476)+4(0.1429+0.0909+0.0667+0.0526+0.0434)\right\}$$

$$= \frac{1}{6}(2.4148)$$

$$= 0.4025 \quad \text{—— (1)}$$

$$\int_0^5 \frac{dx}{4x+5} = \left[\frac{\log(4x+5)}{4}\right]_0^5 = \frac{1}{4}\left[\log 25 - \log 5\right]$$

$$= \frac{1}{4}\log 5 \quad \text{—— (2)}$$

From (1) & (2) $\frac{1}{4}\log 5 = 0.4025$

$\log 5 = 1.61$