Romberg's Integration

Suppose we want to evaluate $I = \int_a^b y\,dx$ by Trapezoidal rule with two different sub interval lengths h_1 & h_2, the approximate value of I are calculated as I_1 & I_2.

Let E_1 & E_2 be the corresponding truncation error for an interval of size h. The error in the trapezoidal rule is given by kh^2, where k is constant.

Then $I = I_1 + E_1 = I_1 + kh_1^2$ —— (1)

$I = I_2 + E_2 = I_2 + kh_2^2$ —— (2)

$\therefore I_1 + kh_1^2 = I_2 + kh_2^2$

$$k = \frac{I_1 - I_2}{h_2^2 - h_1^2}$$

Sub. in (1)

$$I = I_1 + \left(\frac{I_1 - I_2}{h_2^2 - h_1^2}\right) h_1^2$$

$$I = \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2}$$

This I is a better result than either I_1 or I_2.

If $h_1 = h$ & $h_2 = \frac{1}{2}h$, then we get

$$I = \frac{I_1\left(\frac{1}{4}h^2\right) - I_2 h^2}{\frac{h^2}{4} - h^2}$$

$$= \frac{4I_2 - I_1}{3} = I_2 + \frac{1}{3}(I_2 - I_1)$$

$$I = I_2 + \frac{1}{3}(I_2 - I_1) \quad \text{———} \quad (3)$$

We got this result by applying Trapezoidal rule twice. By applying the trapezoidal rule many times, every time halving h, we get a sequence of results $A_1, A_2, A_3 \ldots$ We apply the formula given by (3) to each of adjacent pairs and get the resultants $B_1, B_2, B_3 \ldots$ Again applying the formula given by (3), to each of pairs $B_1, B_2, B_3 \ldots$, we get another sequence of better results $C_1, C_2 \ldots$ Continuing in this way, we proceed until we get two successive values which are very close to each other. This systematic improvement of Richardson's method is called Romberg method (or) Romberg integration.

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Romberg's method correct 4 decimal places. Hence, obtain an approximate value for π.

Soln: To use the method, we shall give various values of h and evaluate the integral.

By taking $h=0.5$, tabulate the values of $y=\frac{1}{1+x^2}$

x :	0	0.5	1
y :	1	0.80	0.50

$$I_1 = \frac{0.5}{2}\left[1.5+2(0.8)\right] = 0.775$$

By taking $h=0.25$, we have the table.

x :	0	0.25	0.5	0.75	1
y :	1	0.9412	0.8	0.64	0.5

$$I_2 = \frac{0.25}{2}\left\{1.5+2(0.9412+0.8+0.64)\right\} = 0.78280$$

By taking $h=0.125$, the tabular values are

x :	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y :	1	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664	0.5

$$I_3 = \frac{0.125}{2}\left\{(1+0.5)+2(0.9846+0.9412+0.8767+0.8+0.7191+0.64+0.5664)\right\}$$

$$= 0.784430$$

Using Romberg's formula for I_1 & I_2, we have

$$I = I_2 + \frac{(I_2 - I_1)}{3}$$

$$= 0.7828 + \left(\frac{0.7828-0.775}{3}\right)$$

$$= 0.7828 + 0.0026$$

$$= 0.78511.$$

Using Romberg's formula for I_2 & I_3 we have

$$I = I_3 + \left(\frac{I_3 - I_2}{3}\right)$$

$$= 0.78473 + \left(\frac{0.78475 - 0.7828}{3}\right)$$

$$= 0.78475 + 0.00015$$

$$= 0.7854$$

$$I = \int_0^1 \frac{dx}{1+x^2} = 0.7854 \quad \text{--- (1)}$$

(ii) By actual integration.

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} 1 = \frac{\pi}{4} \quad \text{--- (2)}$$

From (1) & (2) we have

$$\frac{\pi}{4} = 0.7854$$

Hence $\pi \approx 3.1416$