UNIT - IV

Initial value problems for ordinary Differential Equations.

Single Step (or) pointwise solution:

In solving a differential equation for approximate solution we find numerical values of $y_1, y_2, y_3 \dots$ Corresponding to given values of independent variable values $x_1, x_2, x_3 \dots$ so that the ordered pairs (x_1, y_1) $(x_2, y_2) \dots$ satisfy a particular solution, though approximately. A solution of this type is called a pointwise solution.

Single Step methods (or) pointwise methods.

In these methods we use information about the curve at one point and we do not iterate the solution. The method involves more evaluation of the function. The methods of Taylor series, Euler & Runge-kutta belong to this type.

Multi Step methods (or) step by step methods

These methods required fewer evaluation of the function (past four values of the function) to estimate the solution at a point and iterations are performed till sufficient accuracy is achieved. Estimation of error is possible and the methods are called Predictor - corrector methods.

The methods of Milne & Adam Bashforth belong to this type.

Taylor Series method

Consider the 1st order differential equation

$\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ ——— (1)

The solution of the above initial value problem is obtained in two types.

(i) power series solution &

(ii) pointwise solution.

(i) Power Series solution:

If $y = y(x)$ is the solution of (1), then $y(x)$ can be expanded in a Taylor series about the point $x = x_0$, as

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \cdots$$

$$\Rightarrow y(x) = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \cdots$$

where $y_0^{(r)} = \frac{d^r y}{dx^r}$ at x_0, y_0;

using (1), the derivatives $y_0', y_0'', y_0''' \ldots$ can be found by means of successive differentiation. Expression (2) gives the values of y for every value of x for which (2) converges.

(ii) pointwise Solution.

If $y(x)$ is the solution of (1), then by

Taylor Series

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \cdots$$

Put $x_1 = x_0 + h$

$$y(x_1) = y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \quad (1)$$

Once y_1 has been calculated from (1), y_1', y_1'', y_1''' can be calculated from

$$y' = f(x, y)$$

Expanding $y(x)$ in a Taylor series about $x = x_1$, we get $y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_1''' + \cdots$

where $y_2 = y(x_2)$ & $x_2 = x_1 + h$

The Taylor algorithm is given as follows.

$$y_{m+1} = y_m + \frac{h}{1!} y_m' + \frac{h^2}{2!} y_m'' + \frac{h^3}{3!} y_m''' + \cdots$$

where $y_m^{(r)} = \frac{d^r y}{dx^r}$ at the pt. (x_m, y_m), where $m = 0, 1, 2 \ldots$

Eg 1) Solve $\frac{dy}{dx} = x + y$, given $y(1) = 0$, and get $y(1.1)$, $y(1.2)$ by Taylor series method. Compare your result with the explicit solution.

Soln:

Here $x_0 = 1$, $y_0 = 0$, $h = 0.1$, $y_0 = y(x=1) = 0$.

$y' = (x+y)$	$y_0' = x_0 + y_0 = 1 + 0 = 1$
$y'' = 1 + y'$	$y_0'' = 1 + y_0' = 2$
$y''' = y''$	$y_0''' = y_0'' = 2$
$y^{iv} = y'''$	$y_0^{iv} = y_0''' = 2$ etc

By Taylor series, we have

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots \quad (1)$$

$$\therefore y_1 = y(1.1) = 0 + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(0) + \frac{(0.1)^3}{6}(2) + \cdots \quad (2)$$

$$= 0.1 + 0.01 + 0.0033 + 0.00000833 + 0.000000166 + \cdots$$

$$y(1.1) = 0.11033847.$$

Now, take $x_0 = 1.1$, $h = 0.1$,

$$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{iv} + \cdots \quad (3)$$

We calculate $y_1', y_1'', y_1''', \ldots$ $x_1 = 1.1$, $y_1 = 0.11033847$

$$y_1' = x_1 + y_1 = 1.1 + 0.11033847$$
$$= 1.21033847.$$

$$y_1'' = 1 + y_1' = 2.21033847.$$

$$y_1''' = y_1'' = y_1^{iv} = y_1^{v} = \cdots = 2.21033847.$$

Using in (3)

$$y_2 = y(1.2) = 0.11033847 + \frac{0.1}{1!}(1.21033847) + \frac{(0.1)^2}{2}(2.21033847)$$
$$+ \frac{(0.1)^3}{6}(2.21033847) + \frac{(0.1)^4}{24}(2.21033847) + \cdots$$
$$= 0.11033847 + 0.121033847 + 2.21033847(0.05 + 0.0016\ldots$$

$$= 0.24280160.$$

The exact soln. of $\frac{dy}{dx} = x+y$ is $y = -x-1+2e^{x-1}$

$$y(1.1) = -1.1 - 1 + 2e^{0.1} = 0.11034$$
$$y(1.2) = -1.2 - 1 + 2e^{0.2} = 0.2428.$$

Since we require $f'(x)$ at $x=50$ we use Newton forward formula & to get $f'(x)$ at $x=56$ we use Newton's backward formula.

By Newton's forward formula.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \cdots\right]$$

$$= 1\left[0.0244 - \frac{1}{2}(-0.0003) + \frac{1}{3}(0)\right]$$

$$= 0.02455$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=50} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \cdots\right]$$

$$= 1[-0.0003] = -0.0003$$

By Newton's backward diff formula.

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \left(\frac{dy}{dx}\right)_{v=0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=50} = \frac{1}{1}\left[0.0229 + \frac{1}{2}(-0.0003) + 0\right]$$

$$= 0.02275$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=56} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \cdots\right]$$

$$= \frac{1}{1}[-0.0003]$$

$$= -0.0003$$