Euler's method

In Taylor series method, we obtain approximate solutions of the initial value problem $\frac{dy}{dx} = f(x,y)$ $y(x_0) = y_0$ as a power series in x, & the solution can be used to compute y numerically specified values x near x_0.

In Euler methods, we compute the values of y for $x_i = x_0 + ih$; $i = 1, 2, \dots$ with a step size $h > 0$.

(ie) $y_i = y(x_i)$, where $x_i = x_0 + ih$, $i = 1, 2, 3 \dots$

Euler Method:

Let $y_i = y(x_i)$, where $x_1 = x_0 + h$

Then $y_1 = y(x_0 + h)$. Then by Taylor series,

$$y_1 = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \dots \quad \text{--- (1)}$$

Neglecting the terms with h^2 & higher powers of h, we get from (1)

$$y_1 = y_0 + h f(x_0, y_0) \quad \text{--- (2)}$$

Expression (2) gives an approximate value of y at $x_1 = x_0 + h$.

Ily, we get $y_2 = y_1 + h f(x_1, y_1)$ for $x_2 = x_1 + h$.

$\therefore$ For any n,

$$y_{n+1} = y_n + h f(x_n, y_n), \quad n = 0, 1, 2, \dots$$

(or) $y(x+h) = y(x) + h f(x,y)$ --- (3)

This formula is called Euler's algorithm Error $= O(h^2)$

end of the interval x_0 to x

1. Using Euler's method, solve numerically the equation $y' = x+y$, $y(0) = 1$, for $x = 0.0(0.2)(1.0)$. Check your answer with the exact solution.

Soln:

$f(x,y) = x+y$, $y_0 = 1$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$

$x_4 = 0.8$, $x_5 = 1.0$

By Euler algorithm,

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= y_0 + h[x_0 + y_0]$$
$$= 1 + (0.2)(0+1)$$

(ie) $y(0.2) = 1.2$ //

$$y_2 = y_1 + h f(x_1, y_1)$$
$$= 1.2 + (0.2)[x_1 + y_1]$$
$$= 1.2 + (0.2)[0.2 + 1.2]$$
$$= 1.2 + 0.28$$

$y(0.4) = 1.48$

$$y_3 = y_2 + h f[x_2, y_2]$$
$$= 1.48 + (0.2)[x_2 + y_2]$$
$$= 1.48 + (0.2)[0.4 + 1.48]$$
$$= 1.48 + 0.376$$

$y(0.6) = 1.856$

$$y_4 = 1.856 + 0.2(0.6 + 1.856) = 2.3472$$

$$y_5 = 2.3472 + (0.2)(0.8 + 2.3472)$$

= [illegible] //

$= 2.94664$ //

To find exact soln:

Given $\frac{dy}{dx} - y = x$

$e^{\int P dx} = e^{-x}$, $Q = x$

$\therefore y e^{\int P dx} = \int Q e^{\int P dx} dx + c.$

$y e^{-x} = \int x e^{-x} dx + c. \Rightarrow C = -x - 1 + ce^{x}$

Given $y(0) = 1 \Rightarrow c = 2$

$\therefore y = 2e^{x} - x - 1$

x	0	0.2	0.4	0.6	0.8	1.0
Euler y	1	1.2	1.48	1.856	2.3472	2.94664
Exact y	1	1.2428	1.5836	2.0442	2.6511	3.4366

2. Using E.M find the soln. of the initial value problem $\frac{dy}{dx} = \log(x+y)$, $y(0) = 2$ at $x = 0.2$ by assuming $h = 0.2$.

Soln:

Given $f(x,y) = \log(x+y)$, $x_0 = 0$, $y_0 = 2$, $x_1 = 0.2$, $h = 0.2$.

By Euler's Algorithm,

$y_1 = y_0 + hf(x_0, y_0)$

$= y_0 + h[\log(x_0 + y_0)]$

$= 2 + (0.2)\log(0 + 2)$

$= 2 + 0.2 \log 2$

$= 2 + (0.2)(0.3010)$

$= 2.0602$ //

(ie) $y(0.2) = 2.0602$ //