Finite difference solution of one-D heat equation by implicit & Explicit Method,

Classification of pde of 2nd order.

The most general linear pde of 2nd order can be written as $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$.
where A, B, C, D, E, F are in general func. of x & y.
The above eqn. of 2nd is said to

(i) elliptic if $B^2 - 4AC < 0$
(ii) parabolic if $B^2 - 4AC = 0$
(iii) hyperbolic if $B^2 - 4AC > 0$.

Note: The same diff. eqn. may be elliptic in one region, parabolic in another & hyperbolic in some other region.

For example:

$$xu_{xx} + u_{yy} = 0$$

Here $A = x$, $B = 0$, $C = 1$

$B^2 - 4AC = -4x$

$xu_{xx} + u_{yy} = 0$ is elliptic if $x > 0$,
hyperbolic if $x < 0$
& parabolic if $x = 0$

$xf_{xx} + yf_{yy} = 0$, $x > 0$, $y > 0$, classify the pde.
$A = x$, $B = 0$, $C = y \Rightarrow B^2 - 4AC = -4xy = -ve$
$\therefore$ It is elliptic $\forall x > 0, y > 0$. ($x > 0, y > 0$ given)

Solution of one-D heat equation

Bender-Schmidt's Difference Method (Explicit method)

Consider the one-D heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$.
This is an example of parabolic eqn. where ($\alpha^2 = k/\rho c$)

Setting $\alpha^2 = \frac{1}{a}$, the equation becomes,

$\frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t} = 0$.

To solve this equation by the method of finite differences $u_{xx} = a u_t$ —— (1)

with boundary conditions $u(0,t) = T_0$, $u(l,t) = T_1$ —— (2)

and with initial condition

$$u(x,0) = f(x), \quad 0 < x < l \quad \text{—— (3)}$$

We select a spacing h for the variable x and a spacing k for the time variable t.

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$\& \; u_t = \frac{u_{i,j+1} - u_{i,j}}{k}$$

Substituting these in (1), we have

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = \frac{a}{k}(u_{i,j+1} - u_{i,j}).$$

$$\therefore u_{i,j+1} - u_{i,j} = \frac{k}{ah^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

$$= \lambda(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

where $\lambda = \frac{k}{ah^2}$

(ie) $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$ — (4)

Writing the boundary conditions as

$$u_{0,j} = T_0 \quad \text{— (5)}$$

$$u_{n,j} = T_l \quad \text{— (6)}$$

where $nh = l$

& initial condition as $u_{i,0} = f(ih)$, $i = 1, 2, \ldots$ — (7)

U is known at $t = 0$.

Eqn. (5) facilitates to get the value of u at $x = ih$ & time t_{j+k}.

Eqn. (5) is called Explicit formula. It is valid if $0 < \lambda < 1/2$.

If we take, $\lambda = 1/2$, the coeff. of $u_{i,j}$ vanishes. Hence eqn. (5) becomes.

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right] \quad \text{— (8)}$$

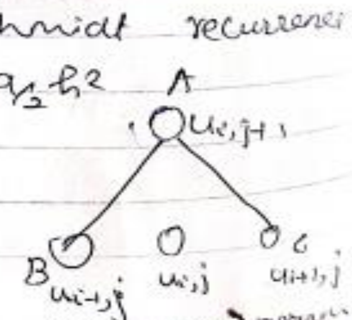
where $\lambda = 1/2 = k/ah^2$, (ie) $k = \frac{a}{2}h^2$

(ie) the value of u at $x = x_i$ at $t = t_{j+1}$ is equal to the average of the values of u the surrounding pts. x_{i-1} & x_{i+1} at previous time t_j.

Eqn. (8) is called Bender-Schmidt recurrence eqn. This is valid only if $k = \frac{a}{2}h^2$

Value of u at A

$= \frac{1}{2}$ [value of u at B + value of u at C]



Schematic diagram

I. State Schmidt's explicit formula for solving heat flow equation.

$$u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda) u_{i,j} + \lambda u_{i-1,j}$$

If $\lambda = \frac{1}{2}$ $\Rightarrow u_{i,j+1} = \frac{1}{2}[u_{i+1,j} + u_{i-1,j}]$.

1. Solve $u_{xx} = 32 u_t$, taking $h = 0.25$ for $t > 0$, $0 < x < 1$ & $u(x,0) = 0$, $u(0,t) = 0$, $u(1,t) = t$.

Solu:

The range for x is $(0,1)$; Given $h = 0.25$

k is not given. For applying Bender-Schmit method

$$k = \frac{a h^2}{2} = \frac{32}{2}\left(\frac{1}{4}\right)^2 = 1$$

step size of time t is 1.

The formula is $u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}]$ — (1)

Using (1), the values of up to $t = 5$ secs are tabulated below

direction of x

$j \backslash i$	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	0.5	2
3	0	0	0.25	1	3
4	0	0.125	0.5	0.625	4
5	0	0.25	0.875	2.25	5

t-direction

values of u.