5 0 0.5 0.75 0.5 0

Crank-Nicholson - Difference method [Implicit method]

Aim: to solve the parabolic equation.

Consider the one-D heat equation

$u_{xx} = a u_t$ with boundary conditions.

$u(0,t) = T_0$, $u(l,t) = T_l$ and the initial condition

$u(x,0) = f(x)$, $0 < x < l$.

The equation to be solved is $u_{xx} = a u_t$ —— (1)

At $u_{i,j}$

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

at $u_{i,j+1}$

$$u_{xx} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}$$

Taking the average of these two values

$$u_{xx} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j+1} + u_{i-1,j}}{2h^2}$$

Using $u_t = \dfrac{u_{i,j+1} - u_{i,j}}{k}$ equ. (1) reduces to

$$\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{2h^2} = a\,\frac{u_{i,j+1} - u_{i,j}}{k}.$$

Setting $\frac{k}{ah^2} = \lambda$, the above equ. reduces to

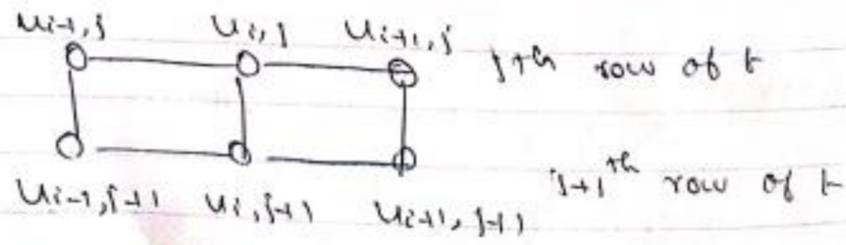
$$\tfrac{1}{2}\lambda u_{i+1,j+1} + \tfrac{1}{2}\lambda u_{i-1,j+1} - (\lambda+1) u_{i,j+1}$$
$$= -\tfrac{1}{2}\lambda u_{i+1,j} - \tfrac{1}{2}\lambda u_{i-1,j} + (\lambda-1) u_{i,j} \quad \text{———} \; \cancel{(\mathrm{I})}$$

This can be written as

$$\lambda(u_{i+1,j+1} + u_{i-1,j+1}) - 2(\lambda+1) u_{i,j+1}$$
$$= 2(\lambda-1) u_{i,j} - \lambda(u_{i+1,j} + u_{i-1,j}) \quad \text{———} \; (\mathrm{I})$$

Equ. I is called Crank - Nicholson difference Scheme (or) Method.

Note: 1. The six points in the above formula are shown below

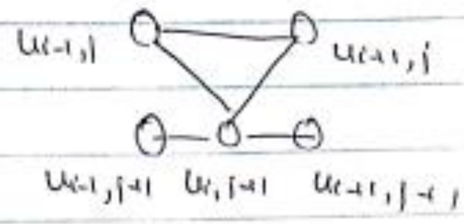


Note: 2

If $\lambda = 1$ (i.e.) $k = ah^2$, then the Crank-Nicholson reduces to

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right]$$

Schematic diagram.



The value of u at A = average of the values at B,C,D

1) Solve by Crank-Nicholson method the equation $u_{xx} = u_t$. Subject to $u(x,0)=0$, $u(0,t)=0$ & $u(1,t)=t$, for two time steps.

Soln.

x ranges from 0 to 1, take $h = 1/4$;

here $a = 1$

$\therefore k = ah^2$ to use simple form

$k = 1\left(\frac{1}{4}\right)^2 = \frac{1}{16}$

We use $u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j}\right]$

Given B.C, $u(0,t)=0$ & $u(1,t)=t$

& Initial condition is $u(x,0)=0$.

x-direction

x - direction

t - direction		0	0.25	0.5	0.75	1
	0	0	0	0	0	0
	$\frac{1}{16}$	0	u_1	u_2	u_3	$\frac{1}{16}$
	$\frac{2}{16}$	0	u_4	u_5	u_6	$\frac{2}{16}$

Let the unknown be represented by u_1, u_2, u_3 ..

$(1) \Rightarrow u_1 = \frac{1}{4}(0+0+0+u_2)$ (ie) $u_1 = \frac{1}{4}u_2$ — (2)

$u_2 = \frac{1}{4}(0+0+u_1+u_3)$ $u_2 = \frac{1}{4}(u_1+u_3)$ — (3)

$u_3 = \frac{1}{4}(0+0+u_2+\frac{1}{16})$ $u_3 = \frac{1}{4}(u_2+\frac{1}{16})$ — (4)

Sub u_1, u_3 values in eqn. (3)

$$u_2 = \frac{1}{4}\left[\frac{1}{4}u_2 + \frac{1}{4}(u_2+\frac{1}{16})\right]$$

$$= \frac{1}{16}\left[u_2+u_2+\frac{1}{16}\right]$$

$$= \frac{1}{16}\left[2u_2+\frac{1}{16}\right]$$

$$= \frac{u_2}{8} + \frac{1}{16}\cdot\frac{1}{16}$$

$$u_2 - \frac{u_2}{8} = \frac{1}{16}\cdot\frac{1}{16}$$

$$\frac{7u_2}{8} = \frac{1}{16}\cdot\frac{1}{16}$$

$$u_2 = \frac{1}{16}\times\frac{1}{16}\times\frac{8}{7} = 0.0045$$

Sub $u_2 = 0.0045$ in eqn. (2)

$u_1 = 0.0011$

Sub $u_2 = 0.0045$ in eqn. (4)

$$u_3 = \frac{1}{4}\left(0.0045 + \frac{1}{16}\right) = 0.0168$$

From table

$u_4 = \frac{1}{4}[0+0+u_2+u_5]$ (i.e.) $u_4 = \frac{1}{4}(0.0045 + u_5)$ — (5)

$u_5 = \frac{1}{4}(u_1+u_3+u_4+u_6)$ $\quad u_5 = \frac{1}{4}(0.0179 + u_4 + u_6)$ — (6)

$u_6 = \frac{1}{4}\left(u_3 + \frac{1}{16} + u_5 + \frac{2}{16}\right)$ $\quad u_6 = \frac{1}{4}(0.192 + u_5)$ — (7)

Sub. u_4, u_6 values in eqn. (6)

$$u_5 = \frac{1}{4}\left(0.0179 + \frac{1}{4}(0.0045 + u_5) + \frac{1}{4}(0.192 + u_5)\right]$$

$$= \frac{1}{4}\times\frac{1}{4}[4\times 0.0179 + 0.0045 + 0.192 + 2u_5]$$

$$= \frac{1}{16}[0.2681 + 2u_5]$$

$$= 0.01675625 + \frac{1}{8}u_5$$

$$u_5 - \frac{1}{8}u_5 = 0.01675625$$

$$u_5 = 0.01675625 \times \frac{8}{7} = 0.01915$$

(5) $\Rightarrow u_4 = \frac{1}{4}(0.0045 + 0.01915)$

$$= 0.00591$$

(7) $\Rightarrow u_6 = \frac{1}{4}(0.192 + 0.01913)$

$$= 0.052787$$

$u_4 = 0.005899$

$u_5 = 0.01913$

$u_6 = 0.05277$