1) Write down the Crank-Nicolson formula to solve $u_t = u_{xx}$. Cor) Write down the implicit form to solve one-D heat flow equation.

Sol:

$$\frac{1}{2}\lambda u_{i+1,j+1} + \frac{1}{2}\lambda u_{i-1,j+1} - (\lambda+1)u_{i,j+1}$$
$$= -\frac{1}{2}\lambda u_{i+1,j} - \frac{1}{2}\lambda u_{i-1,j} + (\lambda-1)u_{i,j}$$

(or) $\lambda(u_{i+1,j+1} + u_{i-1,j+1}) - 2(\lambda+1)u_{i,j+1}$
$= 2(\lambda-1)u_{i,j} - \lambda(u_{i+1,j} + u_{i-1,j})$

2) What type of equations can be solved by Crank-Nicholson's difference formula

Crank-Nicholson's difference formula is used to solve parabolic equations of the form $u_{xx} = au_t$.

2) Using Crank-Nicholson scheme, solve $u_{xx} = 16u_t$, $0<x<1$, $t>0$ given $u(x,0)=0$, $u(0,t)=0$, $u(1,t)=100t$. Compute u for one step in t-direction taking $h=\frac{1}{4}$.

Sol:

Here $a=16$, $h=\frac{1}{4}$

$\therefore k = ah^2$ to use simple form

$k = 16\left(\frac{1}{16}\right) = 1$

$\therefore$ we use $u_{i,j+1} = \frac{1}{4}[u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j}]$

Given B-Cs are $u(0,t)=0$, $u(1,t)=100t$

Given initial condition is $u(x,0)=0$

j/i	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	u_1	u_2	u_3	100

Using (1)

$u_1 = \frac{1}{4}(0+0+0+u_2)$ (or) $u_1 = \frac{1}{4}u_2$ — (1)

$u_2 = \frac{1}{4}(0+0+u_1+u_3)$ $\quad u_2 = \frac{1}{4}(u_1+u_3)$ — (3)

$u_3 = \frac{1}{4}(0+0+u_2+100)$ $\quad u_3 = \frac{1}{4}(u_2+100)$ — (4)

Sub. u_1, u_3 values in (3),

$$u_2 = \frac{1}{4}\left[\frac{1}{4}u_2 + \frac{1}{4}(u_2+100)\right]$$

$$= \frac{1}{16}(u_2+u_2+100)$$

$$= \frac{1}{8}u_2 + \frac{25}{4} \text{ //}$$

$$u_2 - \frac{1}{8}u_2 = \frac{25}{4}$$

$$7/8\, u_2 = 25/4$$

$$u_2 = \frac{25}{4} \times \frac{8}{7} = \frac{50}{7} = 7.1429 \text{ //}$$

Sub. $u_2 = 7.1429$ in (2), $u_1 = 1.7857$

Sub $u_2 = 7.1429$ in (4), $u_3 = 26.7857$

$\therefore u_1 = 1.7857,\ u_2 = 7.1429,\ u_3 = 26.7857$