One dimensional wave equation

[Hyperbolic type]

Introduction

The one dimensional wave equation is of hyperbolic type

Aim: To solve the hyperbolic equation.

Consider the one-D wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

(ie) $a^2 u_{xx} - u_{tt} = 0$ —— (1)

Subject to the initial conditions $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = 0$ with the boundary conditions $u(0,t) = 0$, $u(l,t) = 0$.

Assuming $\Delta x = h$, $\Delta t = k$, we have

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$u_{tt} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

Substituting these values in (1) we get

$$\frac{a^2}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - \frac{1}{k^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = 0.$$

(ie) $\lambda^2 a^2 (u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - u_{i,j+1} + 2u_{i,j} - u_{i,j-1} = 0$

where $\lambda = k/h$.

$$\lambda^2 a^2 u_{i+1,j} - 2\lambda^2 a^2 u_{i,j} + \lambda^2 a^2 u_{i-1,j} - u_{i,j+1} + 2u_{i,j} - u_{i,j-1} = 0$$

$$u_{i,j+1} = 2(1-\lambda^2 a^2) u_{i,j} + \lambda^2 a^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \quad \text{—— (2)}$$

Equ. (2) is called an explicit formula to solve the wave equation.

To get a simpler form, we choose λ such that

$1-\lambda^2 a^2 = 0 \Rightarrow \lambda^2 = \frac{1}{a^2}$

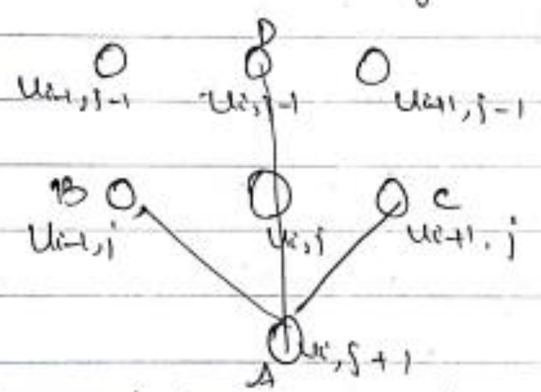
$\therefore \frac{k^2}{h^2} = \frac{1}{a^2} \Rightarrow k = h/a$.

Hence if $k = h/a$ (or) $\lambda^2 = 1/a^2$, the explicit formula (2) takes the form

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \qquad \text{---} (3)$$

Equ. (3) gives a simpler form under the condition $k = h/a$

Note: Schematic diagram



The value of u at A = value of u at (B+C−D)

Solve numerically $4u_{xx} = u_{tt}$ with the boundary conditions, $u(0,t)=0$, $u(4,t)=0$ and the initial conditions $u_t(x,0)=0$ & $u(x,0) = x(4-x)$, taking $h=1$ and up to $t=5$ seconds,

Soln:

Here $a^2=4$, $h=1$.

To use simpler form, take $k = \frac{h}{a} = \frac{1}{2}$. The simplest form of explicit scheme is

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \quad \text{—— (1)}$$

Given Boundary conditions are

$u(0,t) = 0, \forall t$, $\quad u(4,t) = 0, \forall t$

Given initial conditions are,

$u(x,0) = x(4-x)$ —— (2), $\quad u_t(x,0) = 0$ —— (3)

(2) $\Rightarrow u(0,0) = 0,\ u(1,0) = 3,\ u(2,0) = 4,\ u(3,0) = 3,\ u(4,0) = 0$

From (3) $u_t(x,0) = 0 \Rightarrow u_{i,1} = \frac{u_{i+1,0} + u_{i-1,0}}{2}$

put $i = 1, 2, 3$, we have

$$u_{1,1} = \frac{u_{2,0} + u_{1,0}}{2} = \frac{4+0}{2} = 2$$

$$u_{2,1} = \frac{u_{3,0} + u_{1,0}}{2} = \frac{3+3}{2} = 3$$

$$u_{3,1} = \frac{u_{4,0} + u_{2,0}}{2} = \frac{0+4}{2} = 2$$

$u_{4,1} = 0$. Using (1), the values of u are tabulated below.

$j \backslash i$	0	1	2	3	4
0	0	3	4	2	-

j \ i	0	1	2	3	4
0	0	3	4	3	0
0.5	0	2	3	2	0
1	0	0	0	0	0
1.5	0	-2	-3	-2	0
2	0	-3	-4	-3	0
2.5	0	-2	-3	-2	0
3	0	0	0	0	0
3.5	0	2	3	2	0
4	0	3	4	3	0

$u(x,0) = x(4-x)$

Note: $u_t(x,0) = 0 \Rightarrow u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$