Two dimensional Laplace equation

Elliptic equation: $(B^2 - 4AC < 0)$.

The Laplace equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and the poisson equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ are examples of elliptic partial differential equation.

Solution of Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——— (1)}$$

in (1) Replacing the derivatives by finite difference approximations, we get

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = 0.$$

Taking $k = h$, (square mesh) in the above equation, we get

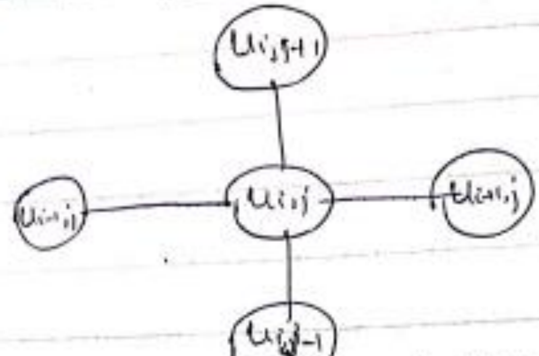
$$4u_{i,j} = u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}$$

$$\therefore u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right] \quad \text{——— (2)}$$

(ie) the value of u at any interior pt. is the arithmetic mean of the values of u vertically at the four lattice pts. (Two of them are ~~fact~~ vertically just above & below and the other two in the horizontal line just after & before the point).

equ. (2) is called Standard five pt. formula

Schematic diagram



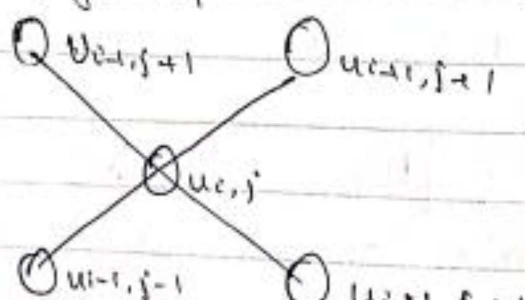
central value = average of the other four values

Diagonal five-pt. formula:

Instead of the formula (3), we can also use the formula.

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right] \quad \text{— (5)}$$

which is called the diagonal five-pt. formula. Since this formula involves the values on the diagonals through $u_{i,j}$



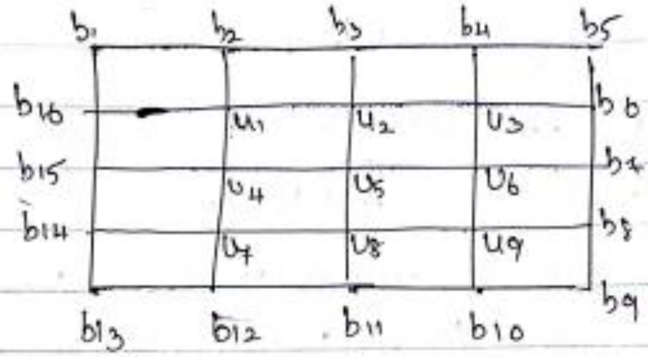
Note:

Since the error in the diagonal five pt. formula is four times the error in the std five pt. formula, we always prefer the std five pt. formula to the diagonal formula.

(By Liebmann's iteration process)

Consider the Laplace's equation $u_{xx}+u_{yy}=0$ in bounded square region R with a boundary C when the boundary values of u are given on the boundary.

Let us divide the square region into a network of sub-squares of side h.



The boundary values of u at the grid pts. are given and noted by $b_1, b_2 \ldots b_{16}$. The values of u at the interior lattice (or) grid pts. are assumed to be $u_1, u_2 \ldots u_9$.

To start the iteration process, initially we find rough values at interior pts. and then we improve them by iterative process mostly using standard five pt. formula.

Find u_5 first $u_5 = \frac{1}{4}(b_3 + b_7 + b_{11} + b_{15})$ (SFPF)

We compute u_1, u_3, u_7, u_9 by using diagonal five pt. formula (DFPF) (ie) $u_1 = \frac{1}{4}(b_3 + b_{15} + b_1 + u_5)$

$u_3 = \frac{1}{4}(b_5 + u_5 + b_3 + b_7)$

$u_7 = \frac{1}{4}(u_5 + b_{13} + b_{11} + b_{15})$

$u_9 = \frac{1}{4}(b_7 + b_{11} + b_9 + u_5)$

The remaining 4 values u_2, u_4, u_6, u_8 can be got by using SFPF

$u_2 = \frac{1}{4}(b_3 + u_5 + u_1 + u_3)$

$u_4 = \frac{1}{4}(u_1 + u_7 + u_5 + b_{15})$

$u_6 = \frac{1}{4}(u_3 + u_9 + u_5 + b_7)$

$u_8 = \frac{1}{4}(u_5 + b_{11} + u_7 + u_9)$

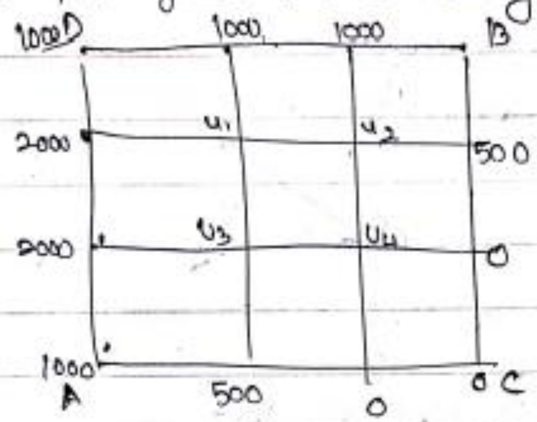
Once all the values $u_1, u_2 \ldots u_9$ are computed their accuracy can be improved by iteration method. The iteration formula is given by

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}\right]$$

where the superscript of u denotes the iteration number.

Equ. (I) is called Liebmann's iteration process. The process is stopped once we get the values with desired accuracy.

Evaluate the fun. $f(x,y)$ satisfying $\nabla^2 u = 0$ at the lattice pts. given the boundary values as follows:



Sol:

We can assume some value for u_4 (or any other u). We can take $u_4 = 0$ and proceed (or) we take a value of $u_4 = 400$ (Given this seeing the values of u on the vertical line through u_2, u_4).

Rough values

$u_1 = \frac{1}{4}(1000 + 2000 + 1000 + u_4)$ DFPF

$\Rightarrow u_1 = \frac{1}{4}(1000 + 2000 + 1000 + 400) = 1100$

$u_2 = \frac{1}{4}(u_1 + u_4 + 1500)$

$= \frac{1}{4}(1100 + 400 + 1500) = 750$ SFPF

$u_3 = \frac{1}{4}(u_1 + 500 + 2000 + u_4)$

$= \frac{1}{4}(1100 + 500 + 2000 + 400) = 1000$ SFPF

$u_4 = \frac{1}{4}(u_2 + u_3)$

$= \frac{1}{4}(750 + 1000) = 437.5$ SFPF.

First iteration

Here after we apply only SFPF

$u_1^{(1)} = \frac{1}{4}(2000+1000+u_2+u_3)$

$= \frac{1}{4}(2000+1000+750+1000)$

$= 1187.5$

$u_2^{(1)} = \frac{1}{4}(1187.5+437.5+1500) = 781.25$

$u_3^{(1)} = \frac{1}{4}(1187.5+437.5+2500) = 1031.25$

$u_4^{(1)} = \frac{1}{4}(781.25+1031.25) = 453.125$

2nd iteration:

$u_1^{(2)} = \frac{1}{4}(781.25+1031.25+3000)$

$= 1203.125$

$u_2^{(2)} = \frac{1}{4}(1203.125+453.125+1500) = 789.1$

$u_3^{(2)} = \frac{1}{4}(1203.125+453.125+2500)$

$= 1039.1$

$u_4^{(2)} = \frac{1}{4}(789.1+1039.1) = 457.1$

3rd iteration:

$u_1^{(3)} = \frac{1}{4}(789.1+1039.1+3000) = 1207.1$

$u_2^{(3)} = \frac{1}{4}(1207.1+457.1+1500) = 791.$

$u_3^{(3)} = \frac{1}{4}(1207.1+457.1+2500) = 1041.1$

$u_4^{(3)} = \frac{1}{4}(791.1+1041.1) = 458.1$

Fourth iteration:

$U_1^{(4)} = \frac{1}{4}(791.1 + 1041.1 + 3000) = 1208.1$

$U_2^{(4)} = \frac{1}{4}(1208.1 + 458.1 + 1500) = 791.6$

$U_3^{(4)} = \frac{1}{4}(1208.1 + 458.1 + 2500) = 1041.6$

$U_4^{(4)} = \frac{1}{4}(791.6 + 1041.6) = 458.3$

Fifth iteration:

$U_1^{(5)} = \frac{1}{4}(791.6 + 1041.6 + 3000) = 1208.3$

$U_2^{(5)} = \frac{1}{4}(1208.3 + 458.3 + 1500) = 791.7$

$U_3^{(5)} = \frac{1}{4}(1208.3 + 458.3 + 2500) = 1041.7$

$U_4^{(5)} = \frac{1}{4}(791.7 + 1041.7) = 458.4$

We are getting result correct to one decimal place. Further the increase in the value is < 0.1

$\therefore u_1 = 1208.1, \; u_2 = 791.7, \; u_3 = 1041.7, \; u_4 = 458.7$