Two dimensional poisson equation

Solution of poisson equation:

An equation of the form $\nabla^2 u = f(x,y)$

(ie) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x,y)$ —— (1)

is called poisson's equation. where $f(x,y)$ is a fun. of x & y only.

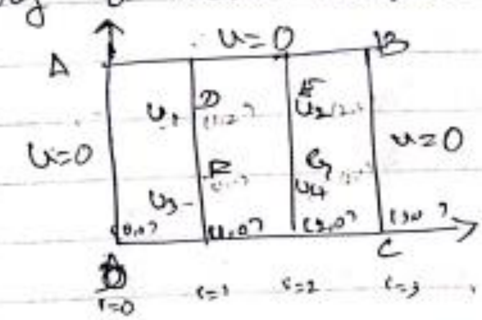
We will solve the above equation numerically over a square mesh, replacing the derivatives by difference quotients. Taking $x = ih$, $y = jk = jh$ the differential equation reduces to

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} = f(ih, jh)$$

(ie) $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)$ —— (2)

By applying the above formula at each mesh point, we get a system of linear equation.

1) Solve $\nabla^2 u = -10(x^2+y^2+10)$ over the square mesh, with sides $x=0$, $y=0$, $x=3$, $y=3$ with $u=0$ on the boundary & mesh length 1 unit.



Soln: Given P.D.E is $\nabla^2 u = -10(x^2+y^2+10)$ — (1)

Given mesh length $\Delta x = h = 1$.

The formula is

$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)$

$\Rightarrow u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(i^2+j^2+10)$ — (2) $\because h=1$

Applying the formula (2) at D $(i=1, j=2)$

$0+0+u_2+u_3-4u_1 = -10(1+4+10) = -150$

$\Rightarrow u_2+u_3-4u_1 = -150$ — (3)

Applying the formula (2) at E $(i=2, j=2)$

$u_1+u_4-4u_2 = -180$ — (4)

Applying the formula (2) at F $(i=1, j=1)$

$u_1+u_4-4u_3 = -120$ — (5)

Applying (2) at G, $(i=2, j=1)$

$u_2+u_3-4u_4 = -10(2^2+1+10) = -150$ — (6)

We can solve the equation (3), (4), (5), (6) either by direct elimination (or) by Gauss-Seidal method.

Method 1:

(5) − (4) gives (Eliminat u_1)

$4(u_2-u_3) = 60$

$u_2-u_3 = 15$ — (7)

Eliminate u_1 from (3) & (4), $3+4(4)$ gives

$-15u_2 + u_3 + 4u_4 = -870$ — (8)

Adding (6) & (8) $-7u_2 + u_3 = -510$ — (9)

From (7), (9) adding $u_2 = 82.5$

Using (7), $u_3 = u_2 - 15 = 82.5 - 15 = 67.5$

Put in (3), $4u_1 = 300$, $\therefore u_1 = 75$

$4u_4 = 150 + 150; \Rightarrow u_4 = 75$

$\therefore u_1 = u_4 = 75$, $u_2 = 82.5$, $u_3 = 67.5$

Method: 2:

We can use Gauss-seidal method to solve

$u_1 = \frac{1}{4}(150 + u_2 + u_3)$

$u_2 = \frac{1}{4}(2u_1 + 180)$

$u_3 = \frac{1}{4}(2u_1 + 120)$

[illegible]	1	2	3	4	5	6	7	8	9	10
$u_4 = u_1$	-	37.5	65.56	72.64	74.41	74.85	74.96	74.99	75	75
u_2	0	63.75	77.79	81.32	82.21	82.43	82.48	82.5	82.5	82.5
u_3	0	48.75	62.78	66.32	67.21	67.43	67.48	67.5	67.5	67.

We get the values after 9 iterations as

$u_1 = 75 = u_4$, $u_2 = 82.5$, $u_3 = 67.5$.