Problem 2

The differential equation of a Physical Phenomenon is given by,

$$\frac{d^2y}{dx^2} + 500x^2 = 0, \quad 0 \le x \le 1$$

Trial function, $y = a_1(x - x^4)$

Boundry Conditions are, $y(0) = 0$, $y(1) = 0$

calculate the value of the Parameter a_1 by the following Methods.

(i) Point collocation (ii) subdomain collocation (iii) Least square (iv) Galerkin

Solution,

$$y = a_1(x - x^4)$$

$x = 0$; $y = 0$

$x = 1$; $y = 0$

Hence, it satisfies the boundry conditions

(i) Point collocation method,

$$y = a_1(x - x^4)$$

$$y = a_1x - a_1x^4$$

$$\frac{dy}{dx} = a_1 - 4a_1x^3$$

$$\frac{d^2y}{dx^2} = -12a_1x^2$$

$$\text{Residual } R = \frac{d^2y}{dx^2} + 500x^2$$

$$R = -12a_1x^2 + 500x^2$$

In Point collocation method set Residual $R = 0$

$$-12a_1x^2 + 500x^2 = 0$$

$$x^2(-12a_1 + 500) = 0$$

$$-12a_1 + 500 = 0$$

$$-12a_1 = -500$$

$$a_1 = \frac{500}{12}$$

$$\boxed{a_1 = 41.66}$$

Substitute a_1 value in Trial function (4)

$$y = a_1(x - x^4) \Rightarrow y = 41.66\,(x - x^4)$$

(ii) Subdomain collocation method :-

$$\int_0^1 R\,dx = 0$$

$$\int_0^1 (-12a_1x^2 + 500x^2)\,dx = 0$$

$$\int_0^1 -12a_1x^2\,dx + \int_0^1 500x^2\,dx = 0$$

$$-12\,a_1\left(\frac{x^3}{3}\right)_0^1 + 500\left(\frac{x^3}{3}\right)_0^1 = 0$$

$$-12\,a_1\left(\frac{1}{3}-0\right) + 500\left(\frac{1}{3}-0\right) = 0$$

$$-4a_1 + 166.67 = 0$$

$$-4a_1 = -166.67$$

$$a_1 = \frac{166.67}{4}$$

$$\boxed{a_1 = 41.66}$$

(iii) Least square's method,

$$I = \int_0^1 R^2\,dx = 0$$

$$\frac{\partial I}{\partial a_1} = \int_0^1 R\,\frac{\partial R}{\partial a_1} = 0$$

$$R = -12a_1 x^2 + 500x^2$$

$$\frac{\partial R}{\partial a_1} = -12x^2$$

$$\frac{\partial I}{\partial a_1} = \int_0^1 (-12a_1 x^2 + 500x^2)(-12x^2)\,dx = 0$$

$$\int_0^1 (144\,a_1 x^4 - 6000\,x^4)\,dx = 0$$

$$\int_0^1 (144\,a_1 x^4)\,dx - \int_0^1 (6000\,x^4)\,dx = 0$$

$$144 a_1 \left(\frac{x^5}{5}\right)_0^1 \overset{(-)}{+} 6000 \left(\frac{x^5}{5}\right)_0^1 = 0$$

$$144 a_1 \left(\frac{1}{5} - 0\right) - 6000 \left(1/5 - 0\right) = 0$$

$$28.8 a_1 - 1200 = 0$$

$$28.8 a_1 = 1200$$

$$a_1 = \frac{1200}{28.8}$$

$$\boxed{a_1 = 41.667}$$

(iv) Galerkin's Method :-

$$\int_0^1 w_i \, R \, dx = 0$$

w_i = weight function = Trial function = $a_1 (x - x^4)$

$$\int_0^1 a_1 (x - x^4) (-12 a_1 x^2 + 500 x^2) \, dx = 0.$$

$$\int_0^1 (a_1 x - a_1 x^4) (-12 a_1 x^2 + 500 x^2) \, dx = 0.$$

$$\int_0^1 (-12 a_1^2 x^3 + a_1 500 x^3 + 12 a_1^2 x^6 - 500 a_1 x^6) \, dx = 0$$

$$\int_0^1 (-12\, a_1^2\, x^3)\, dx + \int_0^1 (a_1\, 500\, x^3)\, dx$$

$$+ \int_0^1 (12\, a_1^2\, x^6)\, dx + \int_0^1 (-500\, a_1 x^6)\, dx = 0$$

$$-12\, a_1^2 \left(\frac{x^4}{4}\right)_0^1 + 500\, a_1 \left(\frac{x^4}{4}\right)_0^1 + 12\, a_1^2 \left(\frac{x^7}{7}\right)_0^1$$

$$- 500\, a_1 \left(\frac{x^7}{7}\right) = 0$$

$$-12\, a_1^2 \left(1/4 - 0\right) + 500\, a_1 \left(1/4 - 0\right) + 12\, a_1^2 \left(1/7 - 0\right)$$

$$- 500\, a_1 \left(1/7 - 0\right) = 0$$

$$-3\, a_1^2 + 125\, a_1 + 1.714\, a_1^2$$

$$- 71.43\, a_1 = 0$$

$$-1.286\, a_1^2 + 53.57\, a_1 = 0$$

$$a_1 (-1.286\, a_1 + 53.57) = 0$$

$$-1.286\, a_1 + 53.57 = 0$$

$$-1.286\, a_1 = -53.37$$

$$a_1 = \frac{53.37}{1.286} = 41.66$$

Substitute a_1 value in Trail function (y)

$$y = a_1 (x - x^4) \Rightarrow y = 41.667\, (x - x^4)$$