SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore – 35

DEPARTMENT OF MATHEMATICS

UNIT – I MULTIPLE INTEGRALS

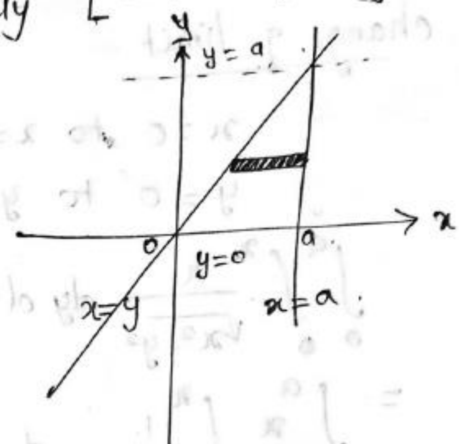
CHANGE OF ORDER OF INTEGRATION

1) Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dy\, dx$ using change of order of Integration.

Given: $\int_{y=0}^{y=a} \int_{x=y}^{x=a} \frac{x}{x^2+y^2}\, \cancel{dy}\, dx\, dy$ [correct form]

Limit $x = y$ to $x = a$

$y = 0$ to $y = a$

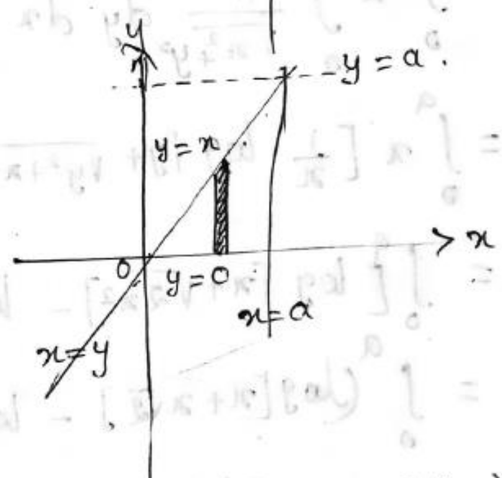


Change of limit:

$y = 0$ to $y = x$

$x = 0$ to $x = a$

$$\int_0^a \int_0^x \frac{x}{x^2+y^2}\, dy\, dx.$$



$$= \int_0^a x \int_0^x \frac{1}{x^2+y^2}\, dy\, dx.$$

$\therefore \int \frac{1}{x^2+y^2}\, dy = \frac{1}{a}\tan^{-1}\left(\frac{y}{a}\right)$

$$= \int_0^a x \cdot \frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right)\Big]_0^x\, dx.$$

$$= \int_0^a \left[\tan^{-1}(1) - \tan^{-1}(0)\right] dx$$

$$= \int_0^a \frac{\pi}{4}\, dx$$

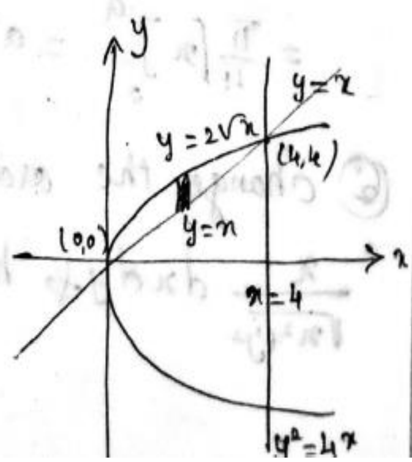
$$= \frac{\pi}{4}\left[x\right]_0^a = a\frac{\pi}{4}.$$

(8) Change of order of integration in $\int_0^4 \int_x^{2\sqrt{x}} x^2+y^2 \, dy\, dx$ & Then evaluate it

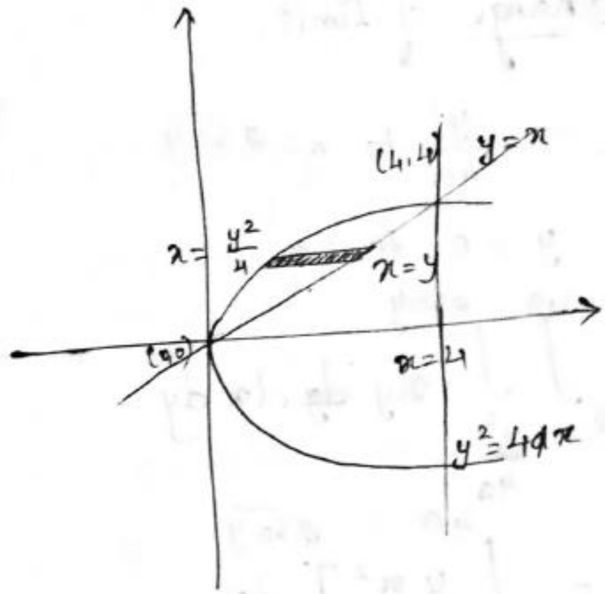
Given: $\int_0^4 \int_x^{2\sqrt{x}} (x^2+y^2)\, dy\, dx$

Limit: $y = x$ to $y = 2\sqrt{x} \Rightarrow y^2 = 4x$

$x = 0$ to $x = 4$



change of Limit



$x = \frac{y^2}{4}$ to $x = y$

$y = 0$ to 4

$$\int_0^4 \int_{\frac{y^2}{4}}^{y} (x^2+y^2)\,dx\,dy$$

$$= \int_0^4 \left[\frac{x^3}{3} + y^2 x\right]_{\frac{y^2}{4}}^{y} dy$$

$$= \int_0^4 \frac{y^3}{3} + y^3 - \left[\frac{y^6}{3\cdot 4^3} + y^2\cdot\frac{y^2}{4}\right] \Big] dy$$

$$= \int_0^4 \left[\frac{4y^3}{3} - \frac{y^6}{3\cdot 4^3} - \frac{y^4}{4}\right] dy$$

$$= \left[\frac{\cancel{4}}{3}\cdot\frac{y^4}{\cancel{4}} - \frac{y^7}{3\cdot 4^3\cdot 7} - \frac{y^5}{4\cdot 5}\right]_0^4$$

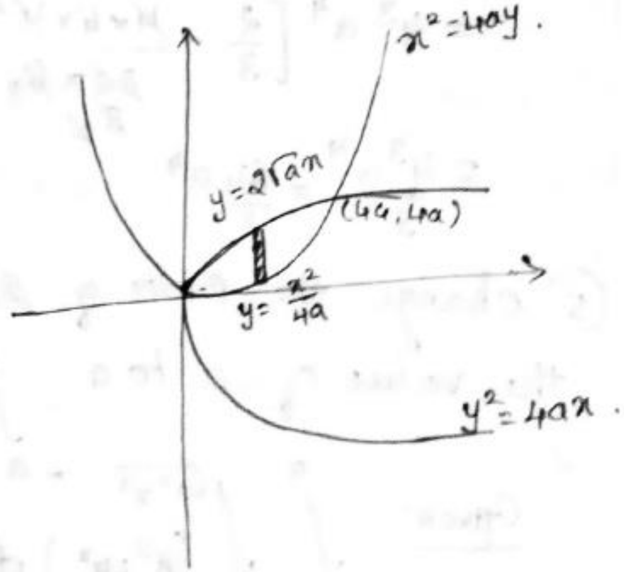
$$= \frac{1}{3}\cdot 4^4 - \frac{4^{\cancel{7}\,4}}{3\cdot \cancel{4^3}\cdot 7} - \frac{4^{\cancel{5}\,4}}{\cancel{4}\cdot 5} = 4^4\left[\frac{1}{3} - \frac{1}{3\cdot 7} - \frac{1}{5}\right]$$

$$= 4^4\left[\frac{35-5-21}{105}\right] = 4^4\,\frac{\cancel{9}^{\,3}}{\cancel{105}_{\,35}} = 4^4\left(\frac{3}{35}\right)$$

4) Change of order of Integration & then evaluate

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx.$$

Given: $$\int_0^{4a}\int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} xy\,dy\,dx.$$



Limit $x=0$ to $x=4a$.

$y=\frac{x^2}{4a}$ to $y=2\sqrt{ax}$

$x^2=4ay$ to $y^2=4ax$

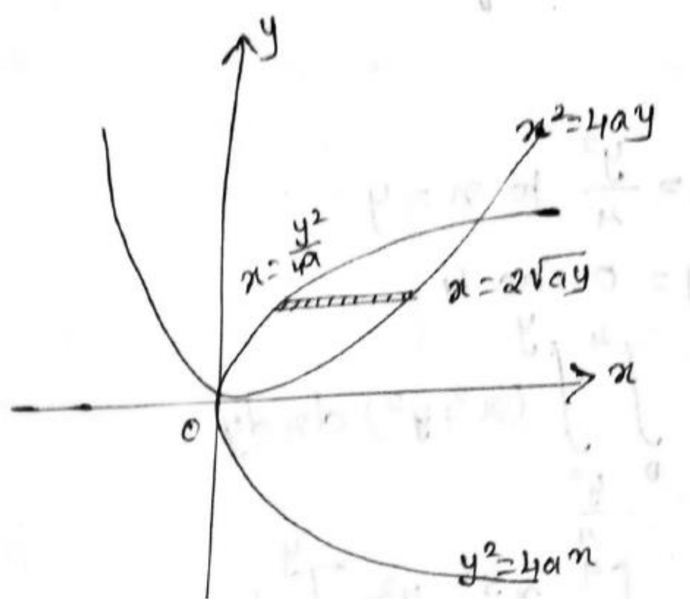
pt of intersection: $(4a,4a)$.

Change of Limit:

$x=\frac{y^2}{4a}$ to $x=2\sqrt{ay}$

$y=0$ to $4a$.

$$\int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} xy\,\cancel{dy}\,dx\,dy$$



$$= \int_0^{4a} y \left.\frac{x^2}{2}\right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a} \frac{y}{2} \left[4ay - \frac{y^4}{16a^2}\right] dy$$

$$= \int_0^{4a} \left[2ay^2 - \frac{y^5}{32a^3}\right] dy$$

$$= \left.\frac{2ay^3}{3} - \frac{y^6}{32.6\,a^3}\right]_0^{4a}$$

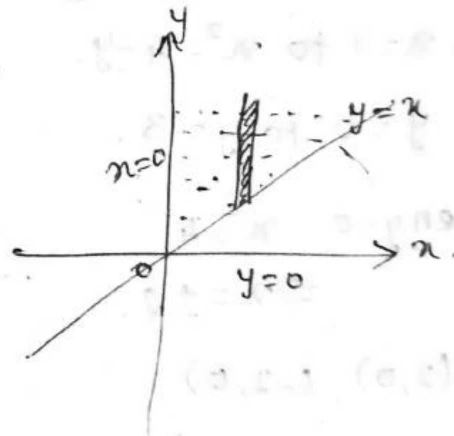
$$= 2a.\frac{4^3a^3}{3} - \frac{4^6 a^{\cancel{6}\,3}}{32.6\,\cancel{a^3}}$$

$$= 4^3 a^4 \left[\frac{2}{3} - \frac{4^3}{32\times 6}\right]$$

$$= 4^3 a^4 \left[\frac{2}{3} - \frac{\cancel{4}\times\cancel{4}\times\cancel{4}\,2}{\cancel{32}_{8\,\cancel{2}}\times\cancel{6}_3}\right].$$

$$= \frac{4^3 a^4}{3} = \frac{64}{3} a^4$$

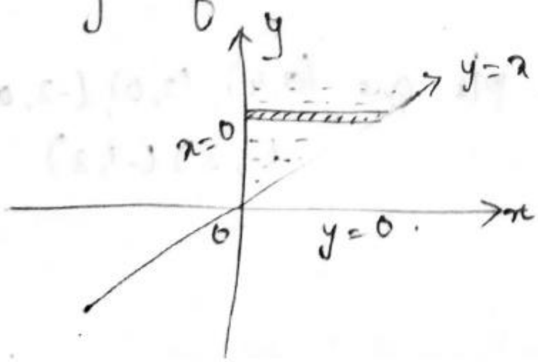
8) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$



$x = 0$ to $x = \infty$

$y = x$ to $y = \infty$

change of limit



Limits:

$x = 0$ to $x = y$

$y = 0$ to $y = \infty$

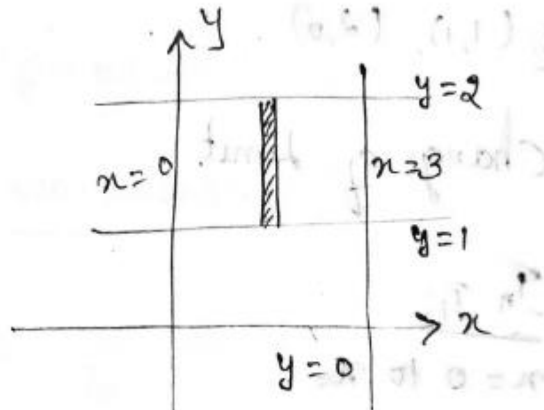
$$\int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \Big[x\Big]_0^y dy = \int_0^\infty e^{-y}\, dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^\infty = \frac{e^{-\infty} - e^0}{-1} = 1.$$

$\begin{cases} e^{-\infty} = 0 \\ e^0 = 1 \end{cases}$

$\int_0^3 \int_1^2 xy(x+y)\, dy\, dx$. Change the order of Integration & evaluate it



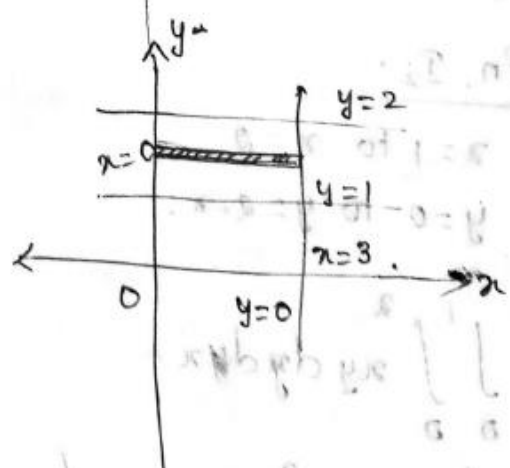
Gn: $x = 0$ to $x = 3$

$y = 1$ to $y = 2$

Change of Limit:

$x = 0$ to $x = 3$

$y = 1$ to $y = 2$



$$\int_1^2 \int_0^3 xy(x+y)\, dx\, dy$$

$$= \int_1^2 \int_0^3 (x^2y + y^2x)\, dx\, dy$$

$$= \int_1^2 \left[y\frac{x^3}{3} + y^2\frac{x^2}{2} \right]_0^3 dy$$

$$= \int_1^2 \left(9y + \frac{9}{2}y^2\right) dy$$

$$= \left[\frac{9y^2}{2} + \frac{9}{2}\frac{y^3}{3} \right]_1^2$$

$$= \frac{9}{2}\left[y^2 + \frac{y^3}{3} \right]_1^2 = \frac{9}{2}\left[4 + \frac{8}{3} - \left(1 + \frac{1}{3}\right) \right]$$

$$= \frac{9}{2}\left[\frac{20}{3} - \frac{4}{3} \right] = \frac{\cancel{9}^{3}}{\cancel{2}} \times \frac{\cancel{16}^{8}}{\cancel{3}}$$

$$= 24.$$