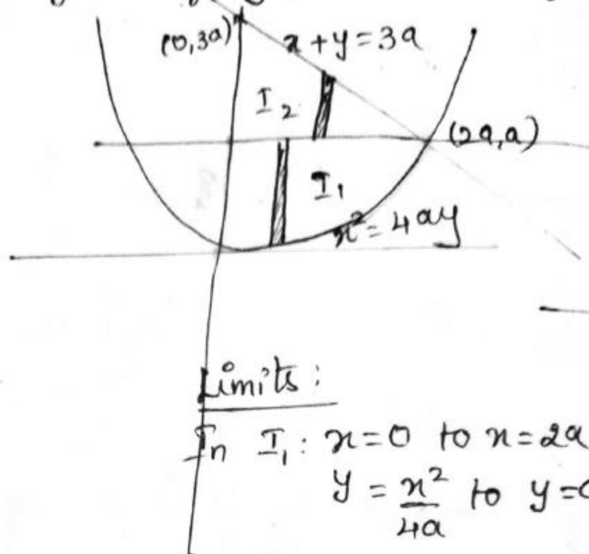
CHANGE OF ORDER OF INTEGRATION

(6) Evaluate $\int_0^{2a}\int_{\frac{x^2}{4a}}^{3a-x} xy\,dy\,dx$ by changing the order of integration.



Given:

$x=0$ to $x=2a$

$y=\frac{x^2}{4a}$ to $y=3a-x$

$\Rightarrow x^2=4ay$ to $x+y=3a$

when $x=0 \Rightarrow y=3a$

$x=2a \Rightarrow y=a$.

$\therefore$ The pt. of intersection $(0,3a)$ & $(2a,a)$

Limits:

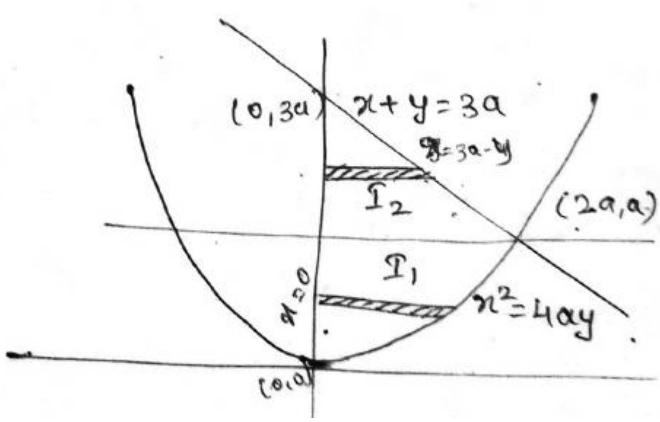
In I_1: $x=0$ to $x=2a$
$y=\frac{x^2}{4a}$ to $y=a$

In I_2: $x=0$ to $x=2a$
$y=a$ to $y=3a-x$

Change of Limit:

In I_1: $x=0$ to $x=2\sqrt{ay}$
$y=0$ to $y=a$

In I_2: $x=0$ to $x=3a-y$
$y=a$ to $y=3a$.



I_1 : $\int_0^a \int_0^{2\sqrt{ay}} xy \, dx\, dy$.

$= \int_0^a y \frac{x^2}{2} \Big]_0^{2\sqrt{ay}} dy$

$= \int_0^a y \frac{4ay}{2} dy = 2a \int_0^a y^2 dy = \frac{2a}{3} y^3 \Big]_0^a = \frac{2}{3} a^4$

I_2 : $\int_a^{3a} \int_0^{3a-y} xy \, dx\, dy$

$= \int_a^{3a} y \frac{x^2}{2} \Big]_0^{3a-y} dy = \frac{1}{2} \int_a^{3a} yx^2 \Big]_0^{3a-y} dy$

$= \frac{1}{2} \int_a^{3a} (3a-y)^2 y\, dy = \frac{1}{2} \int_a^{3a} (9a^2 + y^2 - 6ay) y\, dy = \frac{1}{2} \int_0 (9a^2 y + y^3 - 6ay^2) dy$

$= \frac{1}{2} \Big[9a^2 \frac{y^2}{2} + \frac{y^4}{4} - \cancel{6}^{2} a \frac{y^3}{\cancel{3}} \Big]_a^{3a}$

$= \frac{1}{2} \Big[9a^2 \frac{(3a)^2}{2} + \frac{(3a)^4}{4} - 2a(3a)^3 - \Big[9a^2 \frac{(a)^2}{2} + \frac{a^4}{4} - 2a(a)^3 \Big]\Big]$

$$= \frac{1}{2}\left[\frac{81a^4}{2} + \frac{81}{4}a^4 - 54a^4 - \frac{9a^4}{2} - \frac{a^4}{4} + 2a^4\right]$$

$$= \frac{1}{2}\left[\frac{16}{4}a^4\right] = \frac{4}{2}a^4 = 2a^4.$$

$$I = I_1 + I_2$$

$$= \frac{2}{3}a^4 + 2a^4 = \frac{8}{3}a^4$$

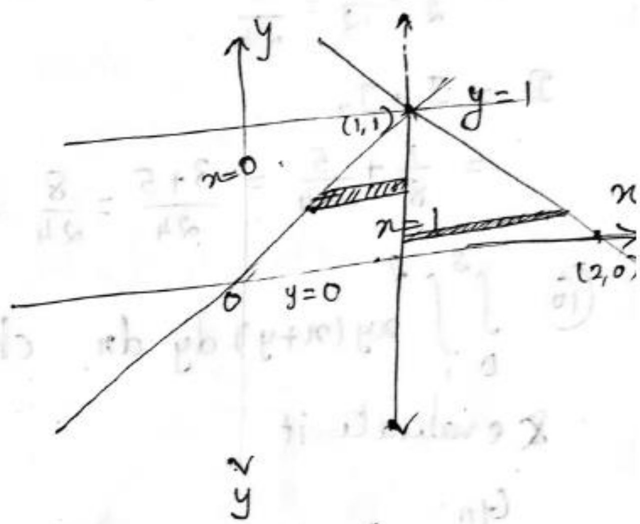
(9) Change the Order of Integration in $\int_0^1 \int_y^{2-y} xy\,dx\,dy$ & hence evaluate it

Gn: $y = 0$ to $y = 1$
$x = y$ to $x = 2-y$

$x = 2-y$

when $y = 0 \Rightarrow x = 2.$
$y = 1 \Rightarrow x = 1$

pts: $(1,1)$, $(2,0)$



Change of Limit

In I_1:

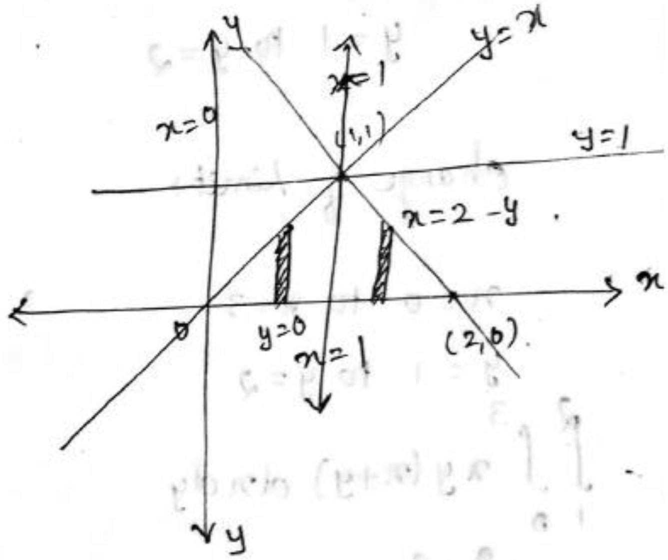
$x = 0$ to $x = 1$

$y = 0$ to $y = x$.

In I_2:

$x = 1$ to $x = 2$

$y = 0$ to $y = 2 - x$.



$$I_1: \int_0^1 \int_0^x xy \, dy \, dx$$

$$= \int_0^1 x \frac{y^2}{2} \Big]_0^x dx = \frac{1}{2} \int_0^1 x^3 \, dx = \frac{1}{2} \cdot \frac{x^4}{4} \Big]_0^1 = \frac{1}{8}$$

$$\int_1^2 \int_0^{2-x} xy \, dy \, dx$$

$$= \int_1^2 \left[x \frac{y^2}{2} \right]_0^{2-x} dx = \frac{1}{2} \int_1^2 x(2-x)^2 \, dx$$

$$= \frac{1}{2} \int_1^2 (4x + x^3 - 4x^2) \, dx$$

$$= \frac{1}{2} \left[4\frac{x^2}{2} + \frac{x^4}{4} - 4\frac{x^3}{3} \right]_1^2$$

$$= \frac{1}{2} \times \frac{5}{12} = \frac{5}{24}.$$

$$I = I_1 + I_2$$

$$= \frac{1}{8} + \frac{5}{24} = \frac{3+5}{24} = \frac{8}{24} = \frac{1}{3}$$