Double integrals:

A double integral ($\iint$) is a way to integrate over a two dimensional area just as an ordinary integral allows you to find the area under the curve, a double integrals helps to find volume under two dimensional area or a surface

Example: 1

$$\int_0^1 \int_0^2 (x+y)\, dy\, dx.$$

Soln:

Given,

$$\int_0^1 \int_0^2 (x+y)\, dy\, dx.$$

$$\Rightarrow \int_0^1 \left[\int_0^2 (x\,dy + y\,dy) \right] dx.$$

$$= \int_0^1 \left[\int_0^2 x\,dy + \int_0^2 y\,dy \right] dx.$$

$$= \int_0^1 \left[[xy]_0^2 + \left[\frac{y^2}{2} \right]_0^2 \right] dx.$$

$$= \int_0^1 \left[[2x - 0(x)] + \left[\frac{2^2}{2} - \frac{0^2}{2} \right] \right] dx.$$

$$= \int_0^1 (2x + 2)\,dx.$$

$$= \int_0^1 2x\,dx + \int_0^1 2\,dx.$$

$$= \left[\frac{2x^2}{2} \right]_0^1 + [2x]_0^1$$

$$= \frac{2}{2} + 2 \quad = 1 + 2$$

$$= 3$$

Example : 2

solve $\int_2^b \int_2^a \frac{dx\,dy}{xy}$ $\quad \frac{1}{xy}\,dx\,dy$

Soln:

Given,

$$\int_2^b \int_2^a \frac{dx\,dy}{xy}$$

$$\Rightarrow \int_2^b \left[\int_2^a \frac{dx}{x} \right] \frac{dy}{y} \quad \frac{1}{x}\,dx$$

$$= \int_2^b \left[[\log x]_2^a \right] \frac{dy}{y}$$

$$= \int_2^b [\log a - \log 2]\, \frac{dy}{y}$$

$$= \int_2^b \log(a/2)\, \frac{dy}{y}$$

$$= \log a/2 \int_2^b \frac{dy}{y}$$

$$= \log\left(\frac{a}{2}\right)\left[\log y\right]_2^b$$

$$= \log(a/2)\left[\log b - \log 2\right]$$

$$= \log(a/2)\log(b/2)$$

$$= \log\left[a + b/2\right]$$

Example : 3

Evalute $\int_0^1 \int_0^{x^2} (x^2 + y^2)\, dy\, dx$.

Soln :

Given,

$$\int_0^1 \int_0^{x^2} (x^2 + y^2)\, dy\, dx$$

$$\Rightarrow \int_0^1 \left[\int_0^{x^2} x^2 dy + \int_0^{x^2} y^2 dy\right] dx. \quad \int dy$$

$$= \int_0^1 \left[(x^2 y)_0^{x^2} + \left[\frac{y^3}{3}\right]_0^{x^2}\right] dx.$$

$$= \int_0^1 \left[(x^4 - 0) + \left[\frac{(x^2)^3}{3} - 0\right]\right] dx.$$

$$= \int_0^1 \left[x^4 + \frac{x^6}{3}\right] dx.$$

$$= \int_0^1 x^4\, dx + \int_0^1 \frac{x^6}{3}\, dx.$$

$$= \left[\frac{x^5}{5}\right]_0^1 + \frac{1}{3}\int_0^1 x^6\, dx.$$

$$= \left[\frac{1}{5} - 0\right] + \frac{1}{3}\left[\frac{x^7}{7}\right]_0^1$$

$$= \frac{1}{5} + \frac{1}{3}\left[\frac{1}{7}\right]$$

$$= \frac{1}{5} + \frac{1}{3}\left(\frac{1}{7}\right)$$

$$= \frac{1}{5} + \frac{1}{21}$$

$$= \frac{21+5}{105}$$

$$= \frac{26}{105}$$

Example : 4

Evalute $\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\, dx$

Soln:

Given,

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\, dx.$$

$$\Rightarrow \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} dy\right] dx.$$

$$= \int_0^a [y]_0^{\sqrt{a^2-x^2}} dx.$$

$$= \int_0^a \sqrt{a^2-x^2}\, dx.$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \left[\frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right] - 0.$$

$$= \frac{a}{2}\sqrt{0} + \frac{a^2}{2}\sin^{-1}(1)$$

$$= \frac{a^2}{2}\sin^{-1}(1)$$

$$= \frac{a^2}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi a^2}{4}$$

Example : 5

Evalute $\int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dy\, dx$

Soln :

Given ,

$$\int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dy\, dx.$$

$$\Rightarrow \int_0^{4a}\left[\int_{y^2/4a}^{2\sqrt{ay}} dx\right] dy.$$

$$= \int_0^{4a}[x]_{y^2/4a}^{2\sqrt{ay}}\, dy$$

$(4)^{3/2} = 4 \cdot 4^{1/2} = 8$

$a^{3/2} = a^{1/2} \cdot a = a\sqrt{a}$

$\sqrt{a} \cdot \sqrt{a} = a.$

$$= \int_0^{4a}\left[2\sqrt{ay} - \frac{y^2}{4a}\right] dy$$

$$= \int_0^{4a}\left[\frac{8a\sqrt{a}\sqrt{y} - y^2}{4a}\right] dy$$

$$= \frac{1}{4a}\int_0^{4a}\left[8a\sqrt{a}\, y^{1/2} - y^2\right] dy$$

$$= \frac{1}{4a}\left[8a\sqrt{a}\int_0^{4a} y^{1/2}\, dy - \int_0^{4a} y^2\, dy\right]$$

$$= \frac{1}{4a}\left[\frac{8a\sqrt{a}\, y^{3/2}}{3/2}\right]_0^{4a} - \left[\frac{y^3}{3}\right]_0^{4a}$$

$$= \frac{1}{4a}\left[\frac{16a\sqrt{a}\,(4a)^{3/2}}{3} - \frac{(4a)^3}{3}\right]$$

$$= \frac{1}{4a}\left[\frac{16 \times (4)^{3/2}\, a\sqrt{a}\; a\sqrt{a}}{3} - \frac{64a^3}{3}\right]$$

$$= \frac{1}{4a}\left[\frac{16 \times 8 \times a^3}{3} - \frac{64a^3}{3}\right]$$

$$= \frac{a^3}{4a}\left[\frac{128}{3} - \frac{64}{3}\right]$$

$$= \frac{a^3}{4a}\left[\frac{64}{3}\right]$$

$$= a^2\left[\frac{16}{3}\right]$$

$$= \frac{16a^2}{3}$$

Example : 6

Evalute $\int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy$

Soln:

Given,

$$\int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy$$

$$\Rightarrow \int_0^a \left[\int_0^{\sqrt{ay}} xy \, dx\right] dy$$

$$= \int_0^a \left[\left[\frac{x^2 y}{2}\right]_0^{\sqrt{ay}}\right] dy$$

$$= \int_0^a \left[\frac{(\sqrt{ay})^2 y}{2} - 0\right] dy$$

$$= \int_0^a \left[\frac{ay^2}{2}\right] dy$$

$$= \int_0^a \frac{ay^2}{2} \, dy$$

$$= \left[\frac{ay^3}{2\times 3}\right]_0^a$$

$$= \frac{aa^3}{6} - 0$$

$$= \frac{a^4}{6}$$

Example : 7

Evalute : $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2) \, dy \, dx.$

Soln:

Given,

$$\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dy\, dx.$$

$$\Rightarrow \int_0^1 \left[\int_x^{\sqrt{x}} (x^2+y^2)\, dy \right] dx.$$

$$= \int_0^1 \left[\int_x^{\sqrt{x}} x^2\, dy + \int_x^{\sqrt{x}} y^2\, dy \right] dx$$

$$= \int_0^1 \left[\left(x^2 y\right)_x^{\sqrt{x}} + \left[\frac{y^3}{3}\right]_x^{\sqrt{x}} \right] dx.$$

$x^2\sqrt{x} = x^2 x^{1/2}$
$= x^{5/2}$
$x\sqrt{x} = x\, x^{1/2}$
$= x^{3/2}$

$$= \int_0^1 \left[\left[x^2\sqrt{x} - x^3\right] + \left[\frac{x\sqrt{x}}{3} - \frac{x^3}{3}\right] \right] dx$$

$$= \int_0^1 \left[x^{5/2} - x^3 + \frac{x^{3/2}}{3} - \frac{x^3}{3} \right] dx$$

$$= \int_0^1 \left[x^{5/2} - x^3 + \frac{x^{3/2}}{3} - \frac{x^3}{3} \right] dx.$$

$$= \int_0^1 x^{5/2} dx - \int_0^1 x^3\, dx + \int_0^1 \frac{x^{3/2}}{3} dx - \int_0^1 \frac{x^3}{3} dx$$

$$= \left[\frac{x^{7/2}}{7/2}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^{5/2}}{5/2 \times 3}\right]_0^1 - \left[\frac{x^4}{4\times 3}\right]_0^1$$

$$= \left[\frac{2x^{7/2}}{7}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2x^{5/2}}{15}\right]_0^1 - \left[\frac{x^4}{12}\right]_0^1$$

$$= \frac{2}{7} - \frac{1}{4} + \frac{2}{15} - \frac{1}{12}$$

$$= \frac{2}{7} + \frac{2}{15} - \left[\frac{1}{4} + \frac{1}{12}\right]$$

$$= \frac{30+14}{105} - \left[\frac{12+4}{48}\right]$$

$$= \frac{44}{105} - \frac{16}{48}$$

$$= \frac{2112 - 1680}{5040}$$

$$= \frac{3}{35}$$

Example: 8

Evalute $\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y \, dx \, dy$

Soln:

Given

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y \, dx \, dy$$

$$\Rightarrow \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} x^2 y \, dy \right] dx$$

$$= \int_0^a \left[\left[\frac{x^2 y^2}{2} \right]_0^{\sqrt{a^2-x^2}} \right] dx$$

$$= \int_0^a \left[\frac{x^2(a^2-x^2)}{2} - 0 \right] dx$$

$$= \int_0^a \left[\frac{x^2 a^2 - x^4}{2} \right] dx$$

$$= \int_0^a \left[\frac{x^2 a^2}{2} - \frac{x^4}{2} \right] dx$$

$$= \int_0^a \frac{x^2 a^2}{2} dx - \int_0^a \frac{x^4}{2} dx$$

$$= \left[\frac{x^3 a^2}{3 \times 2} \right]_0^a - \left[\frac{x^5}{2 \times 5} \right]_0^a$$

$$= \left[\frac{a^2 x^3}{6} \right]_0^a - \left[\frac{x^5}{10} \right]_0^a$$

$$= \left[\frac{a^2 a^3}{6} - 0 \right] - \left[\frac{a^5}{10} - 0 \right]$$

$$= \frac{a^5}{6} - \frac{a^5}{10}$$

$$= \frac{10a^5 - 6a^5}{60}$$

$$= \frac{4a^5}{60}$$

$$= \frac{a^5}{15}$$