Green's Theoream:

Statement:

If R is the closed region of (x, y) bounded by a simply closed curve C, If (M, N) are continous function of (x, y) having continous derivaties in R ie, single integral is,

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Example: 1.

Verify green's theoream and evalute $\int_C (xy + x^2) dx + (x^2 + y^2) dy$ where C is the square formed by $x = -1$, $x = 1$, $y = -1$ and $y = 1$

Soln: Given,

$$\int_C (xy + x^2) dx + (x^2 + y^2) dy.$$

$$M = xy + x^2 \qquad N = x^2 + y^2$$

$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 2x.$$

RHS : $\displaystyle \int_{-1}^{1}\int_{-1}^{1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy.$

$$= \int_{-1}^{1} \left[\int_{-1}^{1} (2x - x)\, dx \right] dy$$

$$= \int_{-1}^{1} \left[2 \int_{-1}^{1} x\, dx - \int_{-1}^{1} x\, dx \right] dy$$

$$= \int_{-1}^{1} \left[2 \int_{-1}^{1} x \, dx - \int_{-1}^{1} x \, dx \right] dy$$

$$= \int_{-1}^{1} \left[2 \left[\frac{x^2}{2} \right]_{-1}^{1} - \left[\frac{x^2}{2} \right]_{-1}^{1} \right] dy$$

$$= \int_{-1}^{1} \left[2 \left[\frac{1}{2} - \frac{1}{2} \right] - \left[\frac{1}{2} - \frac{1}{2} \right] \right] dy$$

$$= \int_{-1}^{1} \left[2(0) - 0 \right] dy$$

$$= \int_{-1}^{1} 0 \, dy$$

$$= 0$$

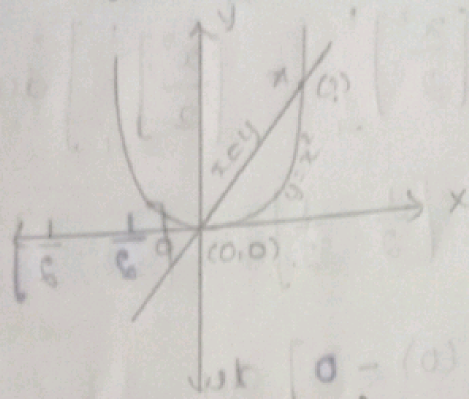
Example: 2

Verify green theorem for $\int_c (xy + y^2)\, dx + x^2\, dy$ where c is the closed region bounded by $y = x^2$, $y = x$

Soln:

Given,

$$\int_C (xy+y^2)\,dx + x^2 dy$$



$y = x^2 \rightarrow ①$
$y = x \rightarrow ②$
sub $y=x$ in ①
$x = x^2$

$x^2 - x = 0$
$x(x-1) = 0$
$x=0 \quad x-1=0$
$x=0 \quad x=1$

If $x=0$ then $y=0$
If $x=1$ then $y=1$

$\frac{\partial M}{\partial y} = 2y + x \qquad \frac{\partial N}{\partial x} = 2x$

RHS :

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_{x^2}^{x} (2x - x - 2y)\,dx\,dy$$

$$= \int_0^1 \int_{x^2}^{x} (x-2y)\,dy\,dx$$

$$= \int_0^1 \left[\int_{x^2}^{x} x\,dy - \int_{x^2}^{x} 2y\,dy\right] dx$$

$$= \int_0^1 \left[(xy)_{x^2}^{x} - 2\left[\frac{y^2}{2}\right]_{x^2}^{x}\right] dx$$

$$= \int_0^1 \left(x^2 - x^3 - 2\left[\frac{x^2}{2} - \frac{x^4}{2}\right]\right] dx$$

$$= \int_0^1 \left[x^2 - x^3 - 2\left[\frac{x^2 - x^4}{2}\right]\right] dx$$

$$= \int_0^1 (x^2 - x^3 - x^2 + x^4)\,dx.$$

$$= \int_0^1 (x^4 - x^3)\,dx$$

$$= \int_0^1 x^4\, dx - \int_0^1 x^3\, dx$$

$$= \left(\frac{x^5}{5}\right)_0^1 - \left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20}$$

$$= \frac{-1}{20}.$$

LHS:

$$\int_C (xy+y^2)\,dx + x^2\,dy = \int_{OA} (xy+y^2)\,dx + x^2\,dy + \int_{AO} (xy+y^2)\,dx + x^2\,dy.$$

Along OA $x^2 = y$

$$2x = \frac{dy}{dx}$$

$$dy = 2x\,dx$$

$$= \int_0^1 (x x^2 + x^4)\,dx + x^2(2x)\,dx$$

$$= \int_0^1 (x^3+x^4)\,dx + 2x^3\,dx$$

$$= \int_0^1 (x^3+x^4)\,dx + \int_0^1 2x^3\,dx.$$

$$= \left[\left[\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 + \left[\frac{2x^4}{4}\right]_0^1\right]$$

$$= \left[\frac{1}{4} + \frac{1}{5}\right] + \frac{1}{2}$$

$$= \frac{9}{20} + \frac{1}{2} = 19/20.$$

Along AO , $x = y$

$$1 = \frac{dy}{dx}$$

$$dx = dy$$

$$\int_{AO} (xy + y^2)\, dx + x^2 dy$$

$$= \int_1^0 (x^2 + x^2)\, dx + x^2 dx$$

$$= \int_1^0 (x^2 + x^2 + x^2)\, dx$$

$$= \int_1^0 3x^2\, dx$$

$$= \left[\frac{3x^3}{3}\right]_1^0$$

$$= -1$$

Now,

$$LHS = \int (xy + y^2)\, dx + x^2 dy + \int (xy + y^2) dx + x^2 dy$$

$$= \frac{19}{20} + (-1)$$

$$= \frac{19}{20} - 1 = -1/20$$

$\therefore$ LHS = RHS

$\therefore$ Hence proved.

Example : 3

Use green's theorem $\int (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$ where c is the curve with $x = 0$, $y = 0$, $x + y = 1$

Soln: Given,

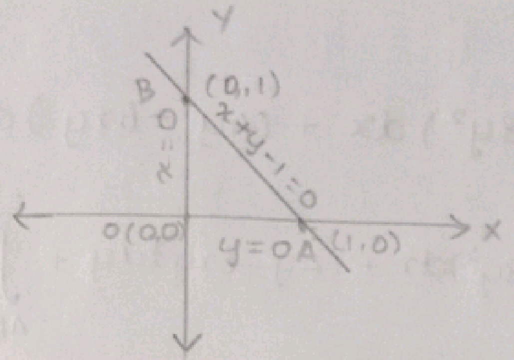
$$\int (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy.$$

$x = 0$, $y = 0$.

$x + y = 1 \quad y = 1 - x$.

~~x~~y varies from 0 to $y = 1 - x$

x varies from 0 to 1



$$M = 3x^2 - 8y^2 \qquad N = 4y - 6xy$$

$$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$$

<u>RHS</u> : $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$$= \int_0^1 \int_0^{1-x} (-6y + 16y)\, dy\, dx$$

$$= \int_0^1 \left[\int_0^{1-x} 10y \ dy\right] dx$$

$$= \int_0^1 \left[10 \left[\frac{y^2}{2}\right]_0^{1-x}\right] dx$$

$$= \int_0^1 \left[\overset{5}{\cancel{10}} \left[\frac{(1-x)^2}{\cancel{2}}\right]\right] dx$$

$$= \int_0^1 \left[5\,(1 + x^2 - 2x)\right] dx$$

$$= \int_0^1 5 + 5x^2 - 10x \ dx$$

$$= 5\left[\frac{x^3}{3}\right]_0^1 - 10\left[\frac{x^2}{2}\right]_0^1 + 5\,(x)_0^1$$

$$= 5\left[\frac{1}{3}\right] - 10\left[\frac{1}{2}\right] + 5$$

$$= \frac{5}{3} - 5 + 5$$

$$= 5/3$$