Gauss divergence Theorem:

If V is the volume enclosed by a closed surface S and If a vector function $\vec{F}$ is continous and It has continous partial derivatives in V on S then,

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \Delta \cdot \vec{F}\, dr.$$

Example:

Verify Gauss Divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$, where $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Soln: Given,

$$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$$

RHS:

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left((x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$\nabla \cdot \vec{F} = 2x + 2y + 2z.$$

$$\iiint_V \nabla \cdot \vec{F}\, dv = 2\int_0^c \int_0^b \left[\int_0^a (x+y+z)\, dx\right] dy\, dz$$

$$= 2\int_0^c \int_0^b \left[\frac{x^2}{2} + xy + zx\right]_0^a dy\, dz$$

$$= 2\int_0^c \left[\int_0^b \left[\frac{a^2}{2} + ay + az\right] dy\right] dz$$

$$= 2\int_0^c \left[\frac{a^2}{2}y + \frac{ay^2}{2} + azy\right]_0^b dz$$

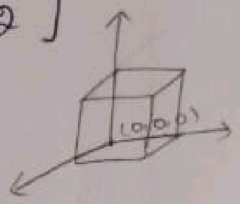
$$= 2\int_0^c \left[\frac{a^2}{2}b + \frac{ab^2}{2} + azb\right] dz$$

$$= 2\left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + \frac{ab\,z^2}{2}\right]_0^c$$

$$= 2\left[\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right]$$

$$= 2abc\left[\frac{(a+b+c)}{2}\right]$$

$$= abc\,(a+b+c)$$



LHS: $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1}\vec{F}\cdot\hat{n}\,ds + \iint_{S_2}\vec{F}\cdot\hat{n}\,ds + \dots + \iint_{S_6}\vec{F}\cdot\hat{n}\,ds$

surfaces	equation	$\hat{n}$	ds
S_1	$x=0$	$-\vec{i}$	$dy\,dz$
S_2	$x=a$	$\vec{i}$	$dy\,dz$
S_3	$y=0$	$-\vec{j}$	$dx\,dz$
S_4	$y=b$	$\vec{j}$	$dx\,dz$
S_5	$y=0$	$-\vec{k}$	$dx\,dy$
S_6	$z=c$	$\vec{k}$	$dx\,dy$.

On S_1.

$$\iint_{S_1}\vec{F}\cdot\hat{n}\,ds = -\int_0^c\int_0^b (x^2 - yz)\,dy\,dz$$

$$= -\int_0^c\int_0^b -yz\,dy\,dz$$

$$= \int_0^c \left[\frac{y^2}{2}\right]_0^b z\,dz$$

$$= \int_0^c \left[\frac{b^2}{2}\right] z\,dz$$

$$= \frac{b^2}{2}\left[\frac{z^2}{2}\right]_0^c$$

$$= \frac{b^2}{2}\times\frac{c^2}{2} = \frac{(bc)^2}{4}$$

On S_2, $x = a$, $\hat{n} = \vec{j}$, $ds = dy\,dz$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b (x^2 - yz)\,dy\,dz$$

$$= \int_0^c \left[\int_0^b (a^2 - yz)\,dy\right] dz$$

$$= \int_0^c \left[a^2y - \frac{y^2z}{2}\right]_0^b dz$$

$$= \int_0^c \left[a^2b - \frac{b^2z}{2}\right] dz$$

$$= \left[a^2bz - \frac{b^2}{2}\times\frac{z^2}{2}\right]_0^c$$

$$= a^2bc - \frac{(bc)^2}{4}$$

On S_3

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^a (y^2 - zx)\,dx\,dz$$

$$= \int_0^c\int_0^a zx\,dx\,dz$$

$$= \int_0^c \left[\frac{zx^2}{2}\right]_0^a dz$$

$$= \int_0^c \frac{za^2}{2}\,dz$$

$$= \int_0^c \frac{a^2}{2}\,z\,dz$$

$$= \left[\frac{a^2}{2}\times\frac{z^2}{2}\right]_0^c \Rightarrow \frac{(ac)^2}{4}$$

On S_4 :

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^a (y^2 - zx)\,dy\,z\,dz$$

$$= \int_0^c\int_0^a (y^2 - zx)\,dx\,dz$$

$$= \int_0^c\int_0^a (b^2 - zx)\,dx\,dz$$

$$= \int_0^c \left[b^2x - \frac{zx^2}{2} \right]_0^a dz$$

$$= \int_0^c \left[ab^2 - \frac{za^2}{2} \right] dz$$

$$= \left[ab^2z - \frac{b^2}{2} \frac{z^2}{2} \right]_0^c$$

$$= \left[ab^2c - \frac{a^2}{2} \times \frac{c^2}{2} \right]$$

$$= ab^2c - \frac{(ac)^2}{4}$$

On $S5$

$$\iint_{S5} \vec{F} \cdot \hat{n}\, ds = \int_0^b \int_0^a (z^2 - xy)\, dx\, dy.$$

$$= \int_0^b \int_0^a xy\, dx\, dy$$

$$= \int_0^b \left[\frac{x^2}{2} y \right]_0^a dy$$

$$= \int_0^b \frac{a^2}{2} y\, dy$$

$$= \frac{a^2}{2} \left[\frac{y^2}{2} \right]_0^b = \frac{a^2}{2} \times \frac{b^2}{2}$$

$$= \frac{(ab)^2}{4}$$

On $S6$

$$\iint_{S6} \vec{F} \cdot \hat{n}\, ds = \int_0^b \int_0^a (z^2 - xy)\, dx\, dy$$

$$= \int_0^b \left[z^2x - \frac{x^2}{2} y \right]_0^a dy$$

$$= \int_0^b \left(c^2a - \frac{a^2 y}{2} \right) dy$$

$$= \int_0^b \left(c^2ay - \frac{a^2y^2}{2 \times 2} \right)_0^b$$

$$= abc^2 - \frac{a^2b^2}{4}$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \frac{\cancel{(bc)^2}}{\cancel{4}} + \frac{a^2bc}{1} - \frac{\cancel{(bc)^2}}{\cancel{4}} + \frac{\cancel{(ac)^2}}{\cancel{4}} + \frac{ab^2c}{1} - \frac{\cancel{(ac)^2}}{\cancel{4}} + \frac{\cancel{(ab)^2}}{\cancel{4}} + \frac{abc^2}{1} - \frac{\cancel{(ab)^2}}{\cancel{4}}$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc\,[a+b+c]$$

2. Verify Gauss theorem $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over region bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$.

Soln:

RHS:

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$$

$$\nabla\cdot\vec{F} = 4z - 2y + y$$

$$\iiint_V \nabla\cdot\vec{F}\, dv = \int_0^1\int_0^1\int_0^1 (4z - 2y + y)\, dx\, dy\, dz$$

$$= \int_0^1\int_0^1 (4zx - 2yx + yx)_0^1\, dy\, dz$$

$$= \int_0^1\int_0^1 (4z - 2y + y)\, dy\, dz$$

$$= \int_0^1 \left(4zy - \frac{y^2}{2}\right)_0^1 dz = \int_0^1 \left(4z - \frac{1}{2}\right) dz$$

$$= \left(\frac{4z^2}{2} - \frac{z}{2}\right)_0^1$$

$$= 2 - \frac{1}{2}$$

$$= 3/2.$$

LHS:

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} \vec{F}\cdot\hat{n}\, ds + \iint_{S_2} \vec{F}\cdot\hat{n}\, ds + \dots + \iint_{S_6} \vec{F}\cdot\hat{n}\, ds$$

Surfaces	equation	$\hat{n}$	ds
S_1	$x=0$	$-\vec{i}$	$dydz$
S_2	$x=1$	$\vec{i}$	$dydz$
S_3	$y=0$	$-\vec{j}$	$dxdz$
S_4	$y=1$	$\vec{j}$	$dxdz$
S_5	$z=0$	$-\vec{k}$	$dxdy$
S_6	$z=1$	$\vec{k}$	$dxdy$

On S_1 $x=0$, $\hat{n}=-\vec{i}$ $ds = dydz$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\, ds = -\iint_{S_1} 4xz\, dydz$$

$$= -\iint_{S_1} 4(0)z\, dydz$$

$$= 0$$

On S_2 $x=1$, $\hat{n}=\vec{i}$ $ds = dydz$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\cdot ds = \iint_{S_2} 4z\, dydz$$

$$= \int_0^1\left(\int_0^1 4z\, dy\right) dz$$

$$= \int_0^1 (4zy)_0^1\, dz$$

$$= \int_0^1 4z\, dz = \left[\frac{4z^2}{2}\right]_0^1$$

$$= 4/2 = 2$$

On S_3, $y=0$, $\hat{n}=-\vec{j}$ $ds = dxdz$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\, ds = -\int_0^1\int_0^1 (-y^2)\, dxdz$$

$$= 0$$

On S_4 : $y=1$, $\hat{n}=\vec{j}$ $ds = dxdz$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 (-y^2)\, dxdz$$

$$= -\int_0^1\int_0^1 dxdz$$

$$= -\int_0^1 (x)_0^1\, dz \quad = -(z)_0^1$$

$$= -1$$

$$= \frac{24}{3} + \frac{54}{3} + \frac{6}{3}$$
$$= 8 + 18 + 2$$
$$= 28.$$

LHS: $\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} \vec{F}\cdot\hat{n}\, ds + \iint_{S_2} \vec{F}\cdot\hat{n}\, ds + \dots + \iint_{S_6} \vec{F}\cdot\hat{n}\, ds$

Surfaces	Equation	$\hat{n}$	ds
S_1	$x=0$	$-\vec{i}$	$dydz$
S_2	$x=1$	$\vec{i}$	$dydz$
S_3	$y=0$	$-\vec{j}$	$dxdz$
S_4	$y=2$	$\vec{j}$	$dxdz$
S_5	$z=0$	$-\vec{k}$	$dxdy$
S_6	$z=3$	$\vec{k}$	$dxdy$

On S_1 $x=0$, $\hat{n} = -\vec{i}$ $ds = dydz$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\, ds = -\int_0^3\int_0^2 xy^2\, dydz$$
$$= 0.$$

On S_2 $x=1$, $\hat{n} = \vec{i}$ $ds = dydz$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\, ds = \int_0^3\int_0^2 y^2\, dydz$$
$$= \int_0^3 \left(\frac{y^3}{3}\right)_0^2 dz$$
$$= \left(\frac{8z}{3}\right)_0^3$$
$$= 24/3$$

On S_3 $y=0$, $\hat{n} = -\vec{j}$ $ds = dxdz$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\, ds = -\int_0^3\int_0^1 yz^2\, dxdz$$
$$= 0$$

On S_4 $y=2$, $\hat{n} = \vec{j}$ $ds = dxdz$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\, ds = \int_0^3\int_0^1 2z^2\, dxdz$$

On S_5 $z=0$, $\hat{n} = -\vec{k}$ $ds = dx\,dy$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = -\int_0^1\int_0^1 yz\,dx\,dy$$

$$= 0$$

S_6 $z=1$, $\hat{n} = \vec{k}$ $ds = dx\,dy$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 y\,dx\,dy$$

$$= \int_0^1 (yx)_0^1\,dy$$

$$= \left(\frac{y^2}{2}\right)_0^1 = \frac{1}{2}$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 0+2+0-1+0+\frac{1}{2}$$

$$= 1+\frac{1}{2} = \frac{3}{2}.$$

2. Verify Gauss divergence theorem $\vec{F} = xy^2\vec{i} + yz^2\vec{j} + zx^3\vec{k}$ over region bounded by $x=0$, $x=1$, $y=0$, $y=2$, $z=0$, $z=3$

Soln:

RHS

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot(xy^2\vec{i} + yz^2\vec{j} + zx^2\vec{k})$$

$$\nabla\cdot\vec{F} = y^2+z^2+z^2$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^3\int_0^2\int_0^1 (y^2+z^2+x^2)\,dx\,dy\,dz$$

$$= \int_0^3\int_0^2 \left(y^2x + z^2x + \frac{x^3}{3}\right)_0^1 dy\,dz$$

$$= \int_0^3\int_0^2 \left(y^2+z^2+\frac{1}{3}\right) dy\,dz$$

$$= \int_0^3 \left(\frac{y^3}{3} + z^2y + y/3\right)_0^2 dz$$

$$= \int_0^3 \left[\frac{8}{3} + 2z^2 + \frac{2}{3}\right] dz$$

$$= \left(\frac{8z}{3} + \frac{2z^3}{3} + \frac{2z}{3}\right)_0^3$$

$$= \int_0^3 (2z^2x)_0^1$$

$$= \left(\frac{2z^3}{3}\right)_0^3$$

$$= 54/3$$

<u>On S_5</u> $z=0$, $\hat{n} = -\vec{k}$ $ds = dxdy$

$$\iint_{S_5} \vec{F} \cdot \hat{n} \cdot ds = \int_0^2\int_0^1 (2x^2)\, dxdy$$

$$= 0$$

<u>On S_6</u> $z = 3$, $\hat{n} = \vec{k}$ $ds = dxdy$

$$\iint_{S_6} \vec{F} \cdot \hat{n}\, ds = \int_0^2\int_0^1 (2x^2)\, dx\, dy$$

$$= \int_0^2 \left(\frac{3x^3}{3}\right)_0^1 dy$$

$$= \int_0^2 dy = (y)_0^2 = 2.$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = 0 + \frac{24}{3} + 0 + \frac{54}{3} + 0 + 2$$

$$= 8 + 18 + 2$$

$$= 28.$$