Transforms of periodic functions:

A function $f(x)$ is said to be periodic if and only if $f(x+p) = f(x)$ is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x will be called the period of the function.

The Laplace transformation of a periodic function $f(t)$ with period p given by,

$$L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt.$$

Problems:

(1) Find the Laplace transform of the rectangular wave given by,

$$f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$$

Soln:

Given: $f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$

$$L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt$$

This fn is periodic in the interval $(0, 2b)$ with period $2b$.

$$L[f(t)] = \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t)\,dt.$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st}\,dt + \int_b^{2b} e^{-st}(-1)\,dt \right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^b - \left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\frac{-1}{s}\left(e^{-st}\right)_0^b + \frac{1}{s}\left(e^{-st}\right)_b^{2b}\right]$$

$$= \frac{1}{s(1-e^{-2bs})}\left[-(e^{-bs}-1) + (e^{-2bs} - e^{-bs})\right]$$

$$= \frac{-e^{-bs}+1+(e^{-bs})^2 - e^{-bs}}{s(1-e^{-2bs})} = \frac{1-2e^{-bs}+(e^{-bs})^2}{s(1-e^{-2bs})}$$

$$= \frac{1}{s(1-e^{-bs})(1+e^{-bs})}(1-e^{-bs})^2$$

$$= \frac{1}{s}\left(\frac{1-e^{-bs}}{1+e^{-bs}}\right)$$

$$= \frac{1}{s}\left(\frac{e^{sb/2} - e^{-sb/2}}{e^{sb/2}+e^{-sb/2}}\right)$$

$$= \frac{1}{s}\tanh\left(\frac{bs}{2}\right).$$

(2) Find the Laplace transform of the half wave rectifier function

$$f(t) = \begin{cases} \sin\omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$$

Soln:

$$L[f(t)] = \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt + 0 \right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-st}}{s^2+\omega^2} (-s \sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-s\pi/\omega} \cdot \omega + \omega}{s^2+\omega^2} \right]$$

$$= \frac{\omega \left[1+\cancel{e^{-s\pi/\omega}} \right]}{(1-e^{-s\pi/\omega})(1+e^{-s\pi/\omega})(s^2+\omega^2)}$$

$$= \frac{\omega}{(1-e^{-s\pi/\omega})(s^2+\omega^2)}$$

③ Find the Laplace transform of

$$f(t) = \begin{cases} t & , 0 < t < a \\ 2a-t & , a < t < 2a \end{cases} \quad \text{with } f(t+2a) = f(t)$$

Soln :

$$L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t) \, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} t \, dt + \int_a^{2a} e^{-st} (2a-t) \, dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[t \left(\frac{e^{-st}}{-s} \right) - \left(\frac{e^{-st}}{s^2} \right) \right]_0^a + \left[(2a-t) \left(\frac{e^{-st}}{-s} \right) - (-1) \left(\frac{e^{-st}}{s^2} \right) \right]_a^{2a} \right.$$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[-t \frac{e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a + \left[-(2a-t) \frac{e^{-st}}{s} + \frac{e^{-st}}{s^2} \right]_a^{2a} \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[\left(\frac{-a e^{-as}}{s} - \frac{e^{-as}}{s^2} \right) - \left(\frac{-1}{s^2} \right) \right] + \right.$$

$$\left. \left[\left(\frac{e^{-2as}}{s^2} \right) - \left(\frac{-a e^{-as}}{s} + \frac{e^{-as}}{s^2} \right) \right] \right.$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{-a e^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{a e^{-as}}{s} - \frac{e^{-as}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1+e^{-2as} - 2e^{-as}}{s^2} \right]$$

$$= \frac{(1-e^{-as})^{\cancel{2}}}{s^2 (1+e^{-as}) \cancel{(1-e^{-as})}}$$

$$= \frac{1-e^{-as}}{s^2 (1+e^{-as})}$$

$$= \frac{1}{s^2} \tanh\left(\frac{as}{2} \right).$$