CONSTRUCTION OF ANALYTIC FUNCTION :

MILNE-THOMSON METHOD:

Let $f(z) = u + iv$ is to constructed :

(i) Suppose the real part u is given, then

$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz$ where

$\phi_1(z,0) = \frac{\partial u}{\partial x}(z,0)$, $\phi_2(z,0) = \frac{\partial u}{\partial y}(z,0)$

(ii) Suppose Imaginary part v is given, then

$f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\,dz$ where

$\phi_1(z,0) = \frac{\partial v}{\partial y}(z,0)$, $\phi_2(z,0) = \frac{\partial v}{\partial x}(z,0)$

(iii) Suppose $(u-v)$ is given.

Let $U = u - v$

Then $f(z) = \frac{F(z)}{1-i}$ where $F(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz + c$

where $\phi_1(z,0) = \frac{\partial u}{\partial x}(z,0)$ & $\phi_2(z,0) = \frac{\partial u}{\partial y}(z,0)$

(iv) Suppose $(u+v)$ is given

Let $V = u + v$

Then $f(z) = \frac{F(z)}{1+i}$ where $F(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\,dz + c$

where $\phi_1(z,0) = \frac{\partial v}{\partial y}(z,0)$ & $\phi_2(z,0) = \frac{\partial v}{\partial x}(z,0)$.

(1) Show that the function $u = x^3 + x^2 - 3xy^2 + 2xy - y^2$ is harmonic and find the corresponding analytic function $f(z) = u + iv$.

Gn. $u = x^3 + x^2 - 3xy^2 + 2xy - y^2$

$u_x = 3x^2 + 2x - 3y^2 + 2y \Rightarrow u_{xx} = 6x + 2$

$u_y = -6xy + 2x - 2y \Rightarrow u_{yy} = -6x - 2$

$\Rightarrow u_{xx} + u_{yy} = 0, \Rightarrow u$ is harmonic.

$f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz$

here $\phi_1(z,0) = \frac{\partial u}{\partial x}(z,0) = 3z^2 + 2z$

$\phi_2(z,0) = \frac{\partial u}{\partial y}(z,0) = 2z$

$\therefore f(z) = \int [(3z^2 + 2z) - i\,2z]\,dz$

$= \frac{3z^3}{3} + \frac{2z^2}{2} - i\,\frac{2z^2}{2} + c$

$f(z) = z^3 + z^2(1 - i) + c$

(2) Find an analytic function whose imaginary part is

$v = e^{2x}[y\cos 2y + x\sin 2y]$

Gn: $v = e^{2x}[y\cos 2y + x\sin 2y]$

$v_x = 2e^{2x}[y\cos 2y + x\sin 2y] + e^{2x}\sin 2y$

$v_y = e^{2x}[\cos 2y + y(-2\sin 2y) + 2x\cos 2y]$

$f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\,dz$

$\phi_1(z,0) = \frac{\partial v}{\partial y}(z,0) = e^{2z}.[1+0+2z] = e^{2z} + 2ze^{2z}$

$\phi_2(z,0) = \frac{\partial v}{\partial x}(z,0) = 2e^{2z}[0] + e^{2z}[0] = 0$

$\therefore f(z) = \int [e^{2z} + 2ze^{2z}]\,dz + c$

$= \int e^{2z}dz + 2\int ze^{2z}dz + c$

$= \frac{e^{2z}}{2} + 2\left[\frac{ze^{2z}}{2} - \frac{e^{2z}}{4}\right] + c$

$= \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2} + c$

$= ze^{2z} + c$

$u = e^x(x\cos y - y\sin y)$

$u_x = \cos y[xe^x + e^x] - y\sin y\, e^x$

$u_y = -xe^x\sin y - e^x[y\cos y + \sin y]$

$u_x(z,0) = ze^z + e^z$

$u_y(z,0) = 0$

$f(z) = \int ze^z + e^z\,dz$

$= ze^z - e^z + e^z$

$= ze^z + c$