BILINEAR TRANSFORMATION:

An expression of the form $w = \frac{az+b}{cz+d}$ where a, b, c, d are complex constants $\ni ad - bc \neq 0$ is called the bilinear transformation.

It is also called as Mobius transformation (or) Linear fractional transformation (or) simply linear transformations.

The expression $(ad-bc)$ is called the determinant of bilinear transformation.

Defn:

If the image of a point z under a transformation $w = f(z)$ is itself, then the point is called a fixed point or an invariant point of the transformation.

The fixed points of the bilinear transformation $w = \frac{az+b}{cz+d}$ are given by $\frac{az+b}{cz+d} = z$.

NOTE: A bilinear transformation has atmost 2 fixed pts.

Defn: The bilinear transformation which maps three distinct points z_1, z_2, z_3 in z plane onto w_1, w_2, w_3 resp. in w plane is given by

$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

① Find the bilinear transf. which maps the pts $z=0, 1, \infty$ onto $w=-5, -1, 3$ resp. what are the invariant pts of this transformation.

Soln:

Here $z_1=0,\ z_2=1,\ z_3=\infty$ &

$w_1=-5,\ w_2=-1,\ w_3=3$

The requ. transf. is given by

$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

$$\frac{(z-z_1)z_3\left(\frac{z_2}{z_3}-1\right)}{(z_1-z_2)z_3\left(1-\frac{z}{z_3}\right)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

$$\frac{(z-0)(0-1)}{(0-1)(1-0)} = \frac{(w+5)(-1-3)}{(-5+1)(3-w)}$$

$$\frac{-z}{-1} = \frac{(w+5)(\cancel{-4})}{(\cancel{-4})(3-w)} \Rightarrow z = \frac{w+5}{3-w}$$

$$\Rightarrow 3z - wz = w+5$$

$$\Rightarrow w = \frac{3z-5}{z+1}$$

To find invariant pts:

put $w = z$

$$\Rightarrow z = \frac{3z-5}{z+1}$$

$$\Rightarrow z^2 + z = 3z - 5$$

$$\Rightarrow z^2 - 2z + 5 = 0$$

$$\Rightarrow z = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1+2i$$

$z = 1 \pm 2i$ are the requ. invariant pts of this transformation.

Find the bilinear transformation which maps $\infty, i, 0$ onto $0, i, \infty$.

Here $z_1 = \infty$, $z_2 = i$, $z_3 = 0$ &

$w_1 = 0$, $w_2 = i$, $w_3 = \infty$

The requ. transf. is given by,

$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

$$\frac{\cancel{z_1}\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{\cancel{z_1}\left(1-\frac{z_2}{z_1}\right)(z_3-z)} = \frac{(w-w_1)\cancel{w_3}\left(\frac{w_2}{w_3}-1\right)}{(w_1-w_2)\cancel{w_3}\left(1-\frac{w}{w_3}\right)}$$

$$\frac{(-1)(i)}{(1)(-z)} = \frac{(w)(-1)}{(-i)(1)}$$

$$\Rightarrow \frac{i}{z} = \frac{w}{i}$$

$$\Rightarrow w = \frac{i^2}{z} = -\frac{1}{z},$$ which is the requ. transf.