LAURENT'S SERIES:

Let c_1 and c_2 be two concentric circles $|z-a| = R_1$ and $|z-a| = R_2$ where $R_2 < R_1$. Let $f(z)$ be analytic inside & on the annular region R between c_1 and c_2. Then for any $z \in R$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where $a_n = \frac{1}{2\pi i} \int_{c_1} \frac{f(z)}{(z-a)^{n+1}} dz,$

$b_n = \frac{1}{2\pi i} \int_{c_2} \frac{f(z)}{(z-a)^{1-n}} dz.$

the integrals being taken in the anticlockwise direction.

This series is called Laurent's series of $f(z)$ about the point $z=a$.

NOTE:

(1) In the Laurent's series of $f(z)$ about $z=a$, the terms containing the positive powers $\left[\text{(i.e.,)} \sum_{n=0}^{\infty} a_n (z-a)^n \right]$ is called 'Regular part'.

(2) In the Laurent's series of $f(z)$ about $z=a$, the terms containing the negative powers $\left[\text{(ie)} \sum_{n=1}^{\infty} b_n (z-a)^{-n} \right]$ is called the 'principal part'.

(3) Expand $f(z) = \dfrac{7z-2}{z(z-2)(z+1)}$ in Laurent's Series if,

(i) $|z|<2$, (ii) $|z|>3$, (iii) $2<|z|<3$, (iv) $1<|z+1|<3$

Soln: Gn: $f(z) = \dfrac{7z-2}{z(z-2)(z+1)}$

Now $\dfrac{7z-2}{z(z-2)(z+1)} = \dfrac{A}{z} + \dfrac{B}{z-2} + \dfrac{C}{z+1}$

$\Rightarrow 7z-2 = A(z-2)(z+1) + Bz(z+1) + C(z-2)z$

when $z=0$, $A=1$

when $z=-1$, $C=-3$

when $z=2$, $B=2$.

$\therefore f(z) = \dfrac{1}{z} + \dfrac{2}{z-2} - \dfrac{3}{z+1}$

(i) $|z|<2 \Rightarrow \left|\dfrac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{-2(1-\frac{z}{2})} - \frac{3}{z+1}$$

$$= \frac{1}{z} - \left(1-\frac{z}{2}\right)^{-1} - 3(z+1)^{-1}$$

$$= \frac{1}{z} - \left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots\right] - 3\left[1 - z + z^2 - z^3 + \cdots\right]$$

(ii) $|z| > 3 \Rightarrow \frac{3}{|z|} < 1$.

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{z\left(1-\frac{2}{z}\right)} - \frac{3}{z\left(1+\frac{1}{z}\right)} = \frac{1}{z} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{3}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z} + \frac{2}{z}\left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots\right] - \frac{3}{z}\left[1 - \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots\right]$$

(iii) $2 < |z| < 3$:

$\frac{1}{|z|}$

$|z| > 2, \quad |z| < 3$

$\frac{2}{|z|}$

$\Rightarrow \frac{2}{|z|} < 1, \quad \left|\frac{z}{3}\right| < 1$

$2 < |z| < 3$

$$\therefore f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{z\left(1-\frac{2}{z}\right)} - \frac{3}{z\left(1+1/z\right)}$$

$$= \frac{1}{z} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{3}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z} + \frac{2}{z}\left[1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots\right] - \frac{3}{z}\left[1 - \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots\right]$$

(iv) $1 < |z+1| < 3$

$|z+1| < 3$

Let $u = z+1 \Rightarrow z = u-1$

$1 < |z+1|,$

$1 < |u| < 3$, $|u|>1$ & $|u|<3$

$\Rightarrow \left|\frac{1}{u}\right| < 1$, $\left|\frac{u}{3}\right| < 1$

$\left|\frac{1}{z+1}\right| < 1$

$$\therefore f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$$

$$= \frac{1}{u\left[1-\frac{1}{u}\right]} + \frac{2}{(-3)\left[1-\frac{u}{3}\right]} - \frac{3}{u}$$

$$= \frac{1}{u}\left[1-\frac{1}{u}\right]^{-1} - \frac{2}{3}\left[1-\frac{u}{3}\right]^{-1} - \frac{3}{u}$$

$$= \frac{1}{u}\left[1+\frac{1}{u}+\left(\frac{1}{u}\right)^2+\cdots\right] - \frac{2}{3}\left[1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\cdots\right] - \frac{3}{u}$$

$$\therefore f(z) = \frac{1}{z+1}\left[1+\left(\frac{1}{z+1}\right)+\left(\frac{1}{z+1}\right)^2+\cdots\right] - \frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\cdots\right] - \frac{3}{z+1}$$

4) Expand $\dfrac{z^2-1}{(z+2)(z+3)}$ in the appropriate series in the region

(i) $2 < |z| < 3$ (ii) $|z| > 3$ [$|z| < 2$]

Soln: Let $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ [Since degree of z is greater in Nr]

$$= \frac{z^2-1}{z^2+5z+6}$$

$$\therefore f(z) = 1 + \frac{-5z-7}{(z+2)(z+3)}$$

Now $$\frac{-5z-7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$

when $z=-3$, $B=-8$

when $z=-2$, $A=3$

$$\therefore f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + 3/(z+2) - 8/(z+3)$$

(i) $2 < |z| < 3$:

$\Rightarrow |z| > 2, \quad |z| < 3$

$\Rightarrow \left|\frac{2}{z}\right| < 1, \quad \left|\frac{z}{3}\right| < 1$

$$\therefore f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)} = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \cdots\right] - \frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \cdots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

(ii) Now $|z|>3 \Rightarrow \left|\frac{3}{z}\right|<1$

$$f(z) = 1+\frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\cdots\right]+\frac{8}{z}\left[1-\frac{3}{z}+\left(\frac{3}{z}\right)^2-\cdots\right]$$

$$= 1+\frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}(-1)^n\left(\frac{3}{z}\right)^n.$$

5) Expand the function $f(z)=\dfrac{z}{(z-1)(z-3)}$ in Laurent series valid in $|z|>3$ and $|z-1|<2$.