PROPERTIES:

Change of scale property:

If $L[f(t)] = F(s)$, then $L[f(at)] = \frac{1}{a} F(s/a)$

Proof:

WKT $L[f(at)] = \int_0^\infty e^{-st} f(at)\, dt$

put $at = x \Rightarrow t = x/a$

$dt = \frac{dx}{a}$

$$\Rightarrow L[f(at)] = \int_0^\infty e^{-s(x/a)} f(x) \frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-s(x/a)} f(x)\, dx$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)x} f(x)\, dx$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)t} f(t)\, dt = \frac{1}{a} F\left(\frac{s}{a}\right)$$

FIRST SHIFTING PROPERTY:

If $L[f(t)] = F(s)$ then

(i) $L[e^{-at} f(t)] = F(s+a)$

(ii) $L[e^{at} f(t)] = F(s-a)$

Proof:

(i) By defn. WKT $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st} [e^{-at} f(t)]\, dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\, dt$$

$$= F(s+a)$$

(ii)

$$L[e^{at} f(t)] = \int_0^\infty e^{-st} [e^{at} f(t)]\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$= F(s-a)$$

SECOND SHIFTING PROPERTY:

If $L[f(t)] = F(s)$ and $g(t) = \begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$

then $L[g(t)] = e^{-as}F(s)$.

LAPLACE TRANSFORMS OF DERIVATIVES:

If $L[f(t)] = F(s)$ then $L[f'(t)] = sF(s) - f(0)$

Corollary:

$$L[f''(t)] = s^2 F(s) - s f(0) - f'(0)$$

LAPLACE TRANSFORM OF INTEGRALS:

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

DERIVATIVE OF LAPLACE TRANSFORM:

(or) LAPLACE TRANSFORM OF $tf(t)$

If $L[f(t)] = F(s)$ then $L[tf(t)] = -\frac{d}{ds}[F(s)]$

Find $L[e^{5t}]$ applying change of scale property.

$$L[e^{t}] = \frac{1}{s-1} = F(s)$$

$$L[e^{5t}] = \frac{1}{5} F(s/5)$$

$$= \frac{1}{5}\left[\frac{1}{s/5 - 1}\right]$$

$$= \frac{1}{5} \cdot \frac{5}{s-5}$$

$$= \frac{1}{s-5}$$

Find $L\{t^3 e^{\pm 3t}\}$ [problems in First shifting property]

$$L\{t^3 e^{\pm 3t}\} = L[e^{\pm 3t} t^3]$$

$$= L[t^3]_{s \to s \mp 3}$$ [By first shifting prop

$$= \left[\frac{3!}{s^{3+1}}\right]_{s \to s \mp 3}$$

$$= \left\{\frac{6}{s^4}\right\}_{s \to s \mp 3}$$

$$= \frac{6}{(s \mp 3)^4}$$

3) Find $L[te^{2t}\sin 3t]$

$$L[te^{2t}\sin 3t] = -\frac{d}{ds}\left\{L[e^{2t}\sin 3t]\right\}$$

$$= -\frac{d}{ds}\left\{L[\sin 3t]_{s\to s-2}\right\}$$

$$= -\frac{d}{ds}\left\{\left(\frac{3}{s^2+9}\right)_{s\to s-2}\right\}$$

$$= -\left\{\frac{0-3(2s)}{(s^2+9)^2}\right\}_{s\to s-2}$$

$$= \left\{\frac{6s}{(s^2+9)^2}\right\}_{s\to s-2}$$

$$= \frac{6(s-2)}{\{(s-2)^2+9\}^2}$$

$$= \frac{6(s-2)}{(s^2-4s+13)^2}$$

(11) Find $L\left[\frac{\sin 3t}{t}\right]$

WKT $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds = \int_s^\infty L[f(t)]\,ds$

Here $L\left[\frac{\sin 3t}{t}\right] = \int_s^\infty L[\sin 3t]\,ds$

$$= \int_s^\infty \left(\frac{3}{s^2+9}\right) ds$$

$$= \int_s^\infty \frac{3}{s^2+3^2}\,ds$$

$$= 3\cdot\frac{1}{3}\cdot\tan^{-1}\left(\frac{s}{3}\right)\Big]_s^\infty \qquad \left\{\because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right\}$$

$$= \tan^{-1}\infty - \tan^{-1}(s/3)$$

$$= \pi/2 - \tan^{-1}(s/3) \qquad \left[\because \tan^{-1}(x)+\cot^{-1}(x) = \pi/2\right]$$

$$= \cot^{-1}(s/3)$$