UNIT III ANALYTIC FUNCTIONS

Part-A

Problem 1 State Cauchy – Riemann equation in Cartesian and Polar coordinates.

Solution:

Cartesian form:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Polar form:

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Problem 2 State the sufficient condition for the function f(z) to be analytic.

Solution:

The sufficient conditions for a function $f(z)=u+iv$ to be analytic at all the points in a region R are

(1) $u_x=v_y,\quad u_y=-v_x$

(2) $u_x,\ u_y,\ v_x,\ v_y$ are continuous functions of x and y in region R.

Problem 3 Show that $f(z)=e^z$ is an analytic Function.

Solution:

$$\begin{aligned} f(z)=u+iv&=e^z\\ &=e^{x+iy}\\ &=e^x e^{iy}\\ &=e^x\left[\cos y+i\sin y\right] \end{aligned}$$

$u=e^x\cos y,\ v=e^x\sin y$

$u_x=e^x cosy,\ v_x=e^x siny$

$u_y=-e^x siny,\ v_y=e^x cosy$

i.e., $u_x=v_y,\ u_y=-v_x$

Hence C-R equations are satisfied.

$\therefore f(z)=e^z$ is analytic.

Problem 4 Find whether $f(z)=\overline{z}$ is analytic or not.

Solution:

Given $f(z)=\overline{z}=x-iy$

i.e., $u=x,\ v=-y$

$$\frac{\partial u}{\partial x}=1,\ \frac{\partial v}{\partial x}=0$$
$$\frac{\partial u}{\partial y}=0,\ \frac{\partial v}{\partial y}=-1$$
$\therefore u_x \neq v_y$

C-R equations are not satisfied anywhere.

Hence $f(z)=\bar{z}$ is not analytic.

Problem 5 State any two properties of analytic functions

Solution:

(i) Both real and imaginary parts of any analytic function satisfy Laplace equation.

i.e., $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ or $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$.

(ii) If $w=u+iv$ is an analytic function, then the curves of the family $u(x,y)=c,$ cut orthogonally the curves of the family $v(x,y)=c$.

Problem 6 Show that $f(z)=|z|^2$ is differentiable at $z=0$ but not analytic at $z=0$.

Solution:

$$\underset{z\to 0}{Lim}\frac{f(z)-f(0)}{z-0}=\underset{z\to 0}{Lim}\frac{|z|^2}{z}=\underset{z\to 0}{Lim}\frac{z\bar{z}}{z}=\underset{z\to 0}{Lim}\,\bar{z}=0$$
$\therefore f(z)$ is differentiable at $z=0$.

Let $z=x+iy$

$\bar{z}=x-iy$

$|z|^2=z\bar{z}=(x+iy)(x-iy)=x^2+y^2$

$f(z)=x^2+y^2+i0$

$u=x^2+y^2,\ v=0$

$u_x=2x,\ v_x=0$

$u_y=2y,\ v_y=0$

The C-R equation $u_x=v_y$ and $u_y=-v_x$ are not satisfied at points other than z = 0.

Therefore $f(z)$ is not analytic at points other than $z=0$. But a function can not be analytic at a single point only. Therefore $f(z)$ is not analytic at z = 0 also.

Problem 7 Determine whether the function $2xy+i(x^2-y^2)$ is analytic.

Solution:

Given $f(z)=2xy+i(x^2-y^2)$

i.e., $u=2xy$, $v=x^2-y^2$

$\frac{\partial u}{\partial x} = 2y, \quad \frac{\partial v}{\partial x} = 2x$

$\frac{\partial u}{\partial y} = 2x, \; \frac{\partial v}{\partial y} = -2y$

$\therefore u_x \neq v_y$ and $u_y \neq -v_x$

C-R equations are not satisfied.

Hence $f(z)$ is not analytic function.

Problem 8 Show that $v = sinhx\,cosy$ is harmonic

Solution:

$v = sinhxcosy$

$\frac{\partial v}{\partial x} = coshxcosy, \; \frac{\partial v}{\partial y} = -sinhxsiny$

$\frac{\partial^2 v}{\partial x^2} = sinhxcosy, \; \frac{\partial^2 v}{\partial y^2} = -sinhycosy$

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = sinhxcosy - sinhycosy = 0$

Hence v is a harmonic function.

Problem 9 Construct the analytic function $f(z)$ for which the real part is $e^x cosy$.

Solution:

$u = e^x \cos y$

$\frac{\partial u}{\partial x} = e^x \cos y$

Assume $\frac{\partial u}{\partial x}(x, y) = \phi_1(z, 0)$

$\therefore \phi_1(z, 0) = e^z$

$\frac{\partial u}{\partial y} = -e^x siny$

Assume $\frac{\partial u(x, y)}{\partial y} = \phi_2(z, 0)$

$\therefore \phi_2(z, 0) = 0$

$$f(z) = \int f'(z)dz = \int \phi_1(z, 0)\,dz - i\int \phi_2(z, 0)\,dz$$

$$= \int e^z\,dz - i\int 0$$

$f(z) = e^z + C$.

Problem 10 Prove that an analytic function whose real part is constant must itself be a constant.

Solution:

Let $f(z)=u+iv$ be an analytic function

$\Rightarrow u_x=v_y,\ u_y=-v_x$.............(1)

Given

$u=c(\text{a constant})$

$u_x=0,\ u_y=0$

$\Rightarrow v_y=0\ \&\ v_x=0\ by(1)$

We know that $f(z)=u+iv$

$f'(z)=u_x+iv_x$

$f'(z)=0+i0$

$f'(z)=0$

Integrating with respect to z, $f(z)=C$

Hence an analytic function with constant real part is constant.

Problem 11 Define conformal mapping

Solution:

A transformation that preserves angle between every pair of curves through a point both in magnitude and sense is said to be conformal at that point.

Problem 12 If $w=f(z)$ is analytic prove that $\frac{dw}{dz}=\frac{\partial w}{\partial x}=-i\frac{\partial w}{\partial y}$ where $w=u+iv$ and prove that $\frac{\partial^2 w}{\partial z\partial \bar{z}}=0$

Solution:

$w=u(x,y)+iv(x,y)$ is an analytic function of z.

As $f(z)$ is analytic we have $u_x=v_y$, $u_y=-v_x$

Now $\frac{dw}{dz}=f'(z)=u_x+iv_x=v_y-iu_y=-i(u_y+iv_y)$

$$=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=-i\left[\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right]$$

$$=\frac{\partial}{\partial x}(u+iv)=-i\frac{\partial}{\partial y}(u+iv)$$

$$=\frac{\partial w}{\partial x}=-i\frac{\partial w}{\partial y}$$

W.K.T. $\frac{\partial w}{\partial \bar{z}}=0$

$\therefore \frac{\partial^2 w}{\partial z\,\partial \bar{z}}=0$

Problem 13 Define bilinear transformation. What is the condition for this to be conformal?

Solution:

The transformation $w=\frac{az+b}{cz+d},\ ad-bc\neq 0$ where a, b, c, d are complex numbers is called a bilinear transformation.

The condition for the function to be conformal is $\frac{dw}{dz}\neq 0$.

Problem 14 Find the invariant points or fixed points of the transformation $w=2-\frac{2}{z}$.

Solution:

The invariant points are given by $z=2-\frac{2}{z}$

i.e., $z=2-\frac{2}{z}$

$$z^2=2z-2$$
$$z^2-2z+2=0$$
$$z=\frac{2\pm\sqrt{4-4(1)(2)}}{2(1)}$$
$$=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm 2i}{2}$$
$$=1\pm i$$

The invariant points are $z=1+i,\ 1-i$

Problem 15 Find the critical points of (i) $w=z+\frac{1}{z}$ (ii) $w=z^3$.

Solution:

(i). Given $w=z+\frac{1}{z}$

For critical point $\frac{dw}{dz}=0$

$$\frac{dw}{dz}=1-\frac{1}{z^2}=0$$

$z=\pm i$ are the critical points

(ii). Given $w = z^3$

$\frac{dw}{dz} = 3z^2 = 0$

$z = 0$

$\therefore\ z = 0$ is the critical point.

Part-B

Problem 1 Determine the analytic function whose real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

Solution:

Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$\phi_1(x, y) = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$

$\phi_1(z, 0) = 3z^2 + 6z$

$\phi_2(x, y) = \frac{\partial u}{\partial y} = 6xy - 6y$

$\phi_2(z, 0) = 0$

By Milne Thomason method

$$f(z) = \int \varphi_1(z, 0)\,dz - i\int \varphi_2(z, 0)\,dz$$

$$= \int (3z^2 + 6z)\,dz - 0$$

$$= 3\frac{z^3}{3} + 6\frac{z^2}{2} + C = z^3 + 3z^2 + C$$

Problem 2 Find the regular function $f(z)$ whose imaginary part is $v = e^{-x}[x\cos y + y\sin y]$

Solution:

$v = e^{-x}(x\cos y + y\sin y)$

$\phi_2(x, y) = \frac{\partial v}{\partial x} = e^{-x}[\cos y] + (x\cos y + y\sin y)(-e^{-x})$

$\phi_2(z, 0) = e^{-z} + (z)(-e^{-z}) = e^{-z} - ze^{-z} = e^{-z}(1 - z)$

$\phi_1(x, y) = \frac{\partial u}{\partial y} = e^{-x}[-x\sin y + y\cos y + \sin y(1)]$

$\phi_1(z, 0) = e^{-z}[0 + 0 + 0] = 0$

By Milne's Thomson Method

$$f(z) = \int \phi_1(z, 0)\,dz + i\int \phi_2(z, 0)\,dz$$

$$= \int 0\, dz + i\int (1-z)e^{-z} dz$$
$$= i\left[(1-z)\left[\frac{e^{-z}}{-1}\right]-(-1)\left[\frac{e^{-z}}{(-1)^2}\right]\right]+C$$
$$= i\left[-(1-z)e^{-z}+e^{-z}\right]+C$$
$$= i\left[-e^{-z}+ze^{-z}+e^{-z}\right]+C = i\left[ze^{-z}\right]+C$$

Problem 3 Determine the analytic function whose real part is $\dfrac{\sin 2x}{\cosh 2y-\cos 2x}$.

Solution:

Given $u = \dfrac{\sin 2x}{\cosh 2y-\cos 2x}$

$$\phi_1(x,y)=\frac{\partial u}{\partial x}=\frac{(\cosh 2y-\cos 2x)(2\cos 2x)-\sin 2x(2\sin 2x)}{(\cosh 2y-\cos 2x)^2}$$

$$\phi_1(z,0)=\frac{(1-\cos 2z)(2\cos 2z)-2\sin^2 2z}{(1-\cos 2z)^2}$$
$$=\frac{(1-\cos 2z)(2\cos 2z)-2(1-\cos^2 2z)}{(1-\cos 2z)^2}$$
$$=\frac{(1-\cos 2z)(2\cos 2z)-2(1-\cos 2z)(1+\cos 2z)}{(1-\cos 2z)^2}$$
$$=\frac{2\cos 2z-2(1+\cos 2z)}{1-\cos 2z}=\frac{2\cos 2z-2-2\ \cos 2z}{1-\cos 2z}$$
$$=\frac{-2}{1-\cos 2z}=-\frac{1}{\left(\frac{1-\cos 2z}{2}\right)}$$
$$=-\frac{1}{\sin^2 z}=-\cos ec^2 z$$

$$\phi_2(x,y)=\frac{\partial u}{\partial y}=\frac{(\cosh 2y-\cos 2x)(0)-\sin 2x[2\sinh 2y]}{(\cosh 2y-\cos 2x)^2}$$
$$=\frac{-2\sin 2x\sinh 2y}{(\cosh 2y-\cos 2x)^2}$$

$$\varphi_2(z,0)=0$$

By Milne's Thomson method

$$f(z)=\int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$
$$=\int -cosec^2 z\, dz - 0 = cot\, z + C$$

Problem 4 Prove that the real and imaginary parts of an analytic function $w = u + iv$ satisfy Laplace equation in two dimensions viz $\nabla^2 u = 0$ and $\nabla^2 v = 0$.

Solution:

Let $f(z) = w = u + iv$ be analytic

To Prove: u and v satisfy the Laplace equation.

i.e., To prove: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Given: $f(z)$ is analytic

$\therefore$ u and v satisfy C-R equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ….. (1)

and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ….. (2)

Diff. (1) p.w.r to x we get $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ ….. (3)

Diff. (2) p.w.r. to y we get $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$ ….. (4)

The second order mixed partial derivatives are equal

i.e., $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$

(3) + (4) $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$

$\therefore$ u satisfies Laplace equation

Diff. (1) p.w.r to y we get $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$ … (5)

Diff. (2) p.w.r. to x we get $\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$ … (6)

(5) + (6) $\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y} = 0$

i.e., $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\therefore$ v Satisfies Laplace equation

Problem 5 If f(z) is analytic, prove that $\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right)|f(z)|^2 = 4.|f'(z)|^2$

Solution:

Let f(z) = u + iv be analytic.

Then $u_x = v_y$ and $u_y = -v_x$ (1)
Also $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$ (2)
Now $|f(z)|^2 = u^2 + v^2$ and $f'(z) = u_x + iv_x$

$$\therefore \frac{\partial}{\partial x}|f(z)|^2 = 2u.u_x + 2v.v_x$$

and $\frac{\partial^2}{\partial x^2}|f(z)|^2 = 2\left[u_x^2 + u.u_{xx} + v_x^2 + v.v_{xx}\right]$ (3)

Similarly $\frac{\partial^2}{\partial y^2}|f(z)|^2 = 2\left[u_y^2 + u.u_{yy} + v_y^2 + v.v_{yy}\right]$ (4)

Adding (3) and (4)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\left[u_x^2 + u_y^2 + u(u_{xx} + u_{yy}) + v_x^2 + v_y^2 + v(v_{xx} + v_{yy})\right]$$

$$= 2\left[u_x^2 + v_x^2 + u(0) + v_x^2 + u_x^2 + v(0)\right]$$

$$= 4\left[u_x^2 + v_x^2\right]$$

$$= 4.\left|f'(z)\right|^2$$

Problem 6 Prove that $\nabla^2 \left|\text{Re}\, f(z)\right|^2 = 2\left|f'(z)\right|^2$

Solution:

Let $f(z) = u + iv$

$$\left|\text{Re}\, f(z)\right|^2 = u^2$$

$$\frac{\partial}{\partial x}\left(u^2\right) = 2uu_x$$

$$\frac{\partial^2}{\partial x^2}\left(u^2\right) = \frac{\partial}{\partial x}\left(2uu_x\right)$$

$$= 2\left[uu_{xx} + u_x u_x\right]$$

$$= 2\left[uu_{xx} + u_x{}^2\right]$$

$$\frac{\partial^2}{\partial y^2}\left(u^2\right) = 2\left[uu_{yy} + u_y^2\right]$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(u^2\right) = 2\left[u\left(u_{xx} + u_{yy}\right) + u_x^2 + u_y^2\right]$$

$$= 2\left[u(0) + u_x^2 + u_y^2\right]$$

$$= 2\left|f'(z)\right|^2$$

Problem 7 Find the analytic function $f(z)=u+iv$ given that

$2u+v=e^x[\cos y-\sin y]$

Solution:

Given $2u+v=e^x[\cos y-\sin y]$

$f(z)=u+iv..............(1)$

$if(z)=iu-v..............(2)$

$(1)\times 2\Rightarrow 2f(z)=2u+i2v...........(3)$

$(3)-(2)\Rightarrow (2-i)f(z)=(2u+v)+i(2v-u)............(4)$

$F(z)=U+iV$

$\therefore 2u+v=U=e^x[\cos y-\sin y]$

$$\phi_1(x,y)=\frac{\partial U}{\partial x}=e^x\cos y-e^x\sin y$$

$\phi_1(z,o)=e^z$

$$\phi_2(x,y)=\frac{\partial u}{\partial x}=-e^x\sin y-e^x\cos y$$

$\phi_2(z,o)=-e^z$

By Milne Thomson method

$F'(z)=\phi_1(z,o)-i\phi_2(z,o)$

$\int F'(z)\,dz=\int e^z dz-i\int -e^z dz$

$F(z)=(1+i)e^z+C-----(5)$

From (4) & (5)

$(1+i)e^z+C=(2-i)f(z)$

$$f(z)=\frac{1+i}{2-i}e^z+\frac{C}{2-i}$$

$$f(z)=\frac{1+3i}{5}e^z+\frac{C}{2-i}$$

Problem 8 Find the Bilinear transformation that maps the points $1+i,\ -i,\ 2-i$ of the z-plane into the points $0,\ 1,\ i$ of the w-plane.

Solution:

Given $z_1=1+i,\ w_1=0$

$z_2=-i,\ w_2=1$

$z_3=2-i,\ w_3=i$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(1-i)}{(0-1)(i-w)}=\frac{[z-(1+i)][-i-(2-i)]}{[(1+i)-(-i)][(2-i)-z]}$$
$$\frac{w(1-i)}{(-1)(i-w)}=\frac{(z-1-i)(-i-2+i)}{(1+i+i)(2-i-z)}$$
$$\frac{w(1-i)}{(w-i)}=\frac{(z-1-i)(-2)}{(1+2i)(2-i-z)}$$
$$\frac{w(1-i)}{(w-i)}=\frac{(-2)(z-1-i)}{(1+2i)(2-i-z)}$$
$$\frac{w}{w-i}=\frac{(-2)}{(1+2i)(1-i)}\frac{(z-1-i)}{(2-i-z)}$$
$$\frac{w}{w-i}=\frac{(-2)}{(1-i+2i+2)}\frac{(z-1-i)}{(2-i-z)}$$
$$\frac{w}{w-i}=\frac{(-2)}{(3+i)}\frac{(z-1-i)}{(2-i-z)}$$
$$\frac{w-i}{w}=\frac{(3+i)(2-i-z)}{(-2)(z-1-i)}$$
$$1-\frac{i}{w}=\frac{(3+i)(2-i-z)}{(-2)(z-1-i)}$$
$$\frac{i}{w}=1-\frac{3+i}{(-2)}\frac{(2-i-z)}{(z-1-i)}$$
$$\frac{i}{w}=1+\frac{3+i}{2}\frac{(2-i-z)}{(z-1-i)}$$
$$\frac{i}{w}=\frac{2(z-1-i)+(3+i)(2-i-z)}{2(z-1-i)}$$
$$\frac{w}{i}=\frac{2(z-1-i)}{2(z-1-i)+(3+i)(2-i-z)}$$
$$w=\frac{2i(z-1-i)}{2(z-1-i)+(3+i)(2-i-z)}$$
$$w=\frac{2i(z-1-i)}{2z-2-2i+6-3i-3z+2i+1-zi}$$
$$w=\frac{2i(z-1-i)}{-z+5-3i-zi}.$$

Problem 9 Prove that an analytic function with constant modulus is constant.

Solution:

Let $f(z)=u+iv$ be analytic

By C.R equations satisfied

i.e., $u_x = v_y,\ u_y = -v_x$

$\therefore f(z) = u + iv$

$\Rightarrow |f(z)| = \sqrt{u^2 + v^2} = C$

$\Rightarrow |f(z)|^2 = u^2 + v^2 = C^2$

$u^2 + v^2 = C^2$...............(1)

Diff (1) with respect to x

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$uu_x + vv_x = 0$.............(2)

Diff (1) with respect to y

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$-uv_x + vu_x = 0$............(3)

$(2)\times u + (3)\times v \Rightarrow (u^2 + v^2)u_x = 0$

$\Rightarrow u_x = 0$

$(2)\times v - (3)\times u \Rightarrow (u^2 + v^2)v_x = 0$

$\Rightarrow v_x = 0$

W.K.T $f'(z) = u_x + iv_x = 0$

$f'(z) = 0$

Integrate w.r.to z

$f(z) = C$

Problem 10 When the function $f(z) = u + iv$ is analytic show that $u(x,y) = C_1$ and $v(x,y) = C_2$ are Orthogonal.

Solution:

If $f(z) = u + iv$ is an analytic function of z, then it satisfies C-R equations

$u_x = v_y,\ u_y = -v_x$

Given $u(x,y) = C_1$.............(1)

$v(x,y) = C_2$.............(2)

By total differentiation

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Differentiate equation (1) & (2) we get $du = 0$, $dv = 0$

$$\therefore \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y} = m_1(say)$$

$$\frac{dy}{dx} = \frac{-\partial v/\partial x}{\partial v/\partial y} = m_2(say)$$

$$\therefore m_1\, m_2 = -\frac{-\partial u/\partial x}{\partial u/\partial y} \times \frac{-\partial v/\partial x}{\partial v/\partial y} \quad (\therefore u_x = v_y\, u_y = -v_x)$$

$$\therefore m_1\, m_2 = -1$$

The curves $u(x, y) = C_1$ and $v(x, y) = C_2$ cut orthogonally.

Problem 11 Show that the function $u = \frac{1}{2}\log\left(x^2 + y^2\right)$ is harmonic and determine its conjugate.

Solution:

Given $u = \frac{1}{2}\log\left(x^2 + y^2\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{2}.\frac{1}{x^2 + y^2}(2x) = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}.\frac{1}{x^2 + y^2}(2y) = \frac{y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(x^2 + y^2\right) - x(2x)}{\left(x^2 + y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(x^2 + y^2\right)(1) - 2y^2}{\left(x^2 + y^2\right)^2} = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2 - x^2 + x^2 - y^2}{\left(x^2 + y^2\right)^2} = 0$$

Hence u is harmonic function

To find conjugate of u

$$\phi_1(x, y) = \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}$$

$$\phi_1(z, o) = \frac{1}{z}$$

$\phi_2(x,y)=\frac{\partial u}{\partial y}=\frac{y}{x^2+y^2}$

$\phi_2(z,o)=0$

By Milne Thomson Methods

$f'(z)=\phi_1(z,o)-i\phi_2(z,o)$

$\int f'(z)\,dz=\int\frac{1}{z}dz+0$

$\qquad=\log z+c$

$f(z)=\log\ re^{i\theta}$

$f(z)=u+iv=\log r+i\theta$

$u=\log r$, $v=\theta$

$u=\log\sqrt{x^2+y^2}$ $\quad\left[\therefore r^2=x^2+y^2,\ \ \theta=\tan^{-1}\left(\frac{y}{x}\right)\right]$

$v=\tan^{-1}\left(\frac{y}{x}\right)$ $\therefore$ Conjugate of u is $\tan^{-1}\left(\frac{y}{x}\right)$.

Problem 12 Find the image of the infinite strips $\frac{1}{4}<y<\frac{1}{2}$ under the transformation $w=\frac{1}{z}$.

Solution: $w=\frac{1}{z}\Rightarrow z=\frac{1}{w}$

$z=\frac{1}{u+iv}=\frac{u-iv}{u^2+v^2}$

$\Rightarrow x=\frac{u}{u^2+v^2}$........(1)

$y=-\frac{v}{u^2+v^2}$........(2)

Given strip is $\frac{1}{4}<y<\frac{1}{2}$ when $y=\frac{1}{4}$

$\frac{1}{4}=-\frac{v}{u^2+v^2}$ (by 2)

$u^2+(v+2)^2=4$..........(3)

which is a circle whose centre is at $(0,-2)$ in the w-plane and radius 2.

When $y=\frac{1}{2}$

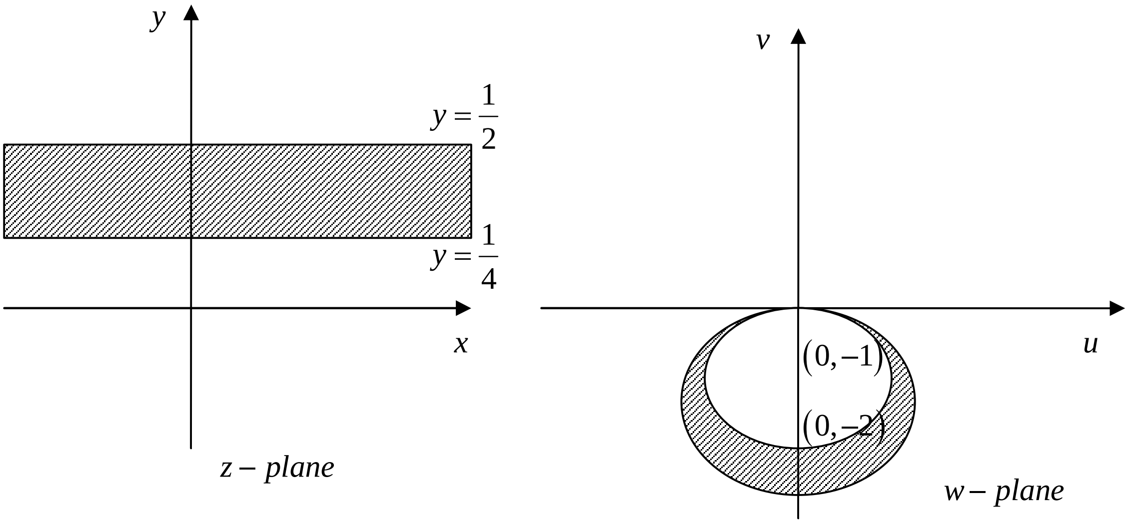
$\frac{1}{2}=\frac{-v}{u^2+v^2}$ (by 2)

$u^2+v^2+2v=0$

$u^2+(v+1)^2=1$..........(4)

which is a circle whose centre is at $(0,-1)$ and radius is 1 in the w-plane.

Hence the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region between circles $u^2 + (v+1)^2 = 1$ and $u^2 + (v+2)^2 = 4$ in the w-plane.



Problem 13 Obtain the bilinear transformation which maps the points $z = 1,\ i,\ -1$ into the points $w = 0, 1, \infty$.

Solution: We know that

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(1-\infty)}{(0-1)(\infty-w)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$\frac{w}{-1}(-1) = \frac{z-1}{1-i} \cdot \frac{i+1}{-(1+z)}$$

$$w = -\frac{z-1}{z+1} \cdot \frac{1+i}{1-i}$$

$$w = (-i)\frac{z-1}{z+1}$$

Problem 14 Find the image of $|z-2i| = 2$ under the transform $w = \frac{1}{z}$

Solution:

Given $w = \frac{1}{w} \Rightarrow z = \frac{1}{w}$

Now $w = u + iv$

$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$$

i.e., $x+iy = \dfrac{u-iv}{u^2+v^2}$

$\therefore \; x = \dfrac{u}{u^2+v^2}$ ....(1)

$y = \dfrac{-v}{u^2+v^2}$ ....(2)

Given $|z-2i| = 2$

$|x+iy-2i| = 2$

$|x+i(y-2)| = 2$

$x^2+(y-2)^2 = 4$

$x^2+y^2-4y = 0$..................(3)

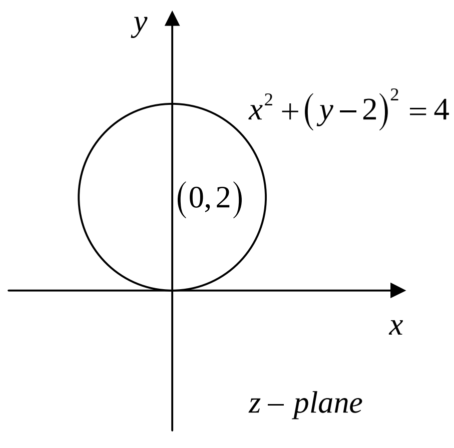
Sub (1) and (2) in (3)

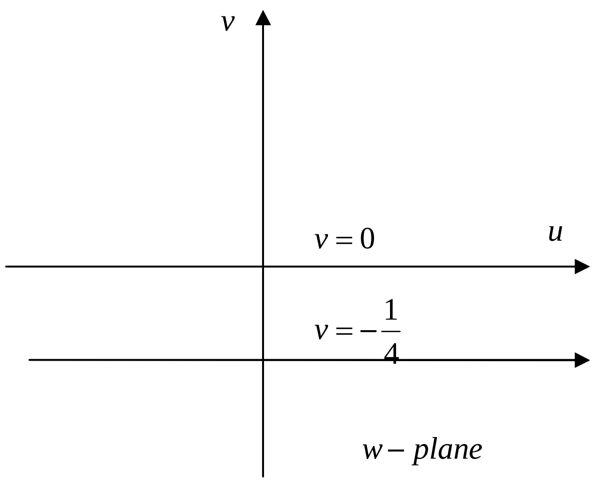
$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 4\left[\frac{-v}{u^2+v^2}\right] = 0$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + 4\left[\frac{4v}{u^2+v^2}\right] = 0$$

$$\frac{(u^2+v^2)+4v(u^2+v^2)}{(u^2+v^2)^2} = 0 \qquad \frac{(1+4v)(u^2+v^2)}{(u^2+v^2)^2} = 0$$

$1+4v = 0 \Rightarrow v = -\dfrac{1}{4}$ $\quad (\therefore u^2+v^2 \neq 0)$ This is straight line in w-plane.





Problem 15 Prove that $w=\frac{z}{1-z}$ maps the upper half of the z-plane onto the upper half of the w-plane.

Solution:

$$w=\frac{z}{1-z}\Rightarrow w(1-z)=z$$

$$w-wz=z$$

$$w=\left(w+1\right)z$$

$$w=(w+1)z$$

$$z=\frac{w}{w+1}$$

Put $z=x+iy,\quad w=u+iv$

$$x+iy=\frac{u+iv}{u+iv+1}$$

$$=\frac{\left(u+iv\right)\left(u+1\right)-iv}{\left(u+iv+1\right)\left(u+1\right)-iv}$$

$$=\frac{u\left(u+1\right)-iuv+iv\left(u+1\right)+v^2}{\left(u+1\right)^2+v^2}$$

$$=\frac{\left(u^2+v^2+u\right)+iv}{\left(u+1\right)^2+v^2}$$

Equating real and imaginary parts

$$x=\frac{u^2+v^2+u}{\left(u+1\right)^2+v^2},\ y=\frac{v}{\left(u+1\right)^2+v^2}$$

$$y=0\Rightarrow\frac{v}{\left(u+1\right)^2+v^2}=0$$

$$y>0\Rightarrow\frac{v}{\left(u+1\right)^2+v^2}>0\Rightarrow v>0$$

Thus the upper half of the z plane is mapped onto the upper half of the w plane.