# UNIT V LAPLACE TRANSFORM

## Part – A

Problem 1 State the conditions under which Laplace transform of $f(t)$ exists

Solution:

(i) $f(t)$ must be piecewise continuous in the given closed interval $[a,b]$ where $a>0$ and

(ii) $f(t)$ should be of exponential order.

Problem 2 Find (i) $L\left[t^{3/2}\right]$ (ii) $L\left[e^{-at}\cos bt\right]$

Solution:

(i) We know that

$$L\left[t^n\right]=\frac{\Gamma(n+1)}{s^{n+1}}$$

$$L\left[t^{n/3}\right]=\frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}}=\frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{5/2}} \quad \left[\because \Gamma(n+1)=n\Gamma(n)\right]$$

$$=\frac{\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)}{s^{5/2}}$$

$$=\frac{\frac{3}{2}\cdot\frac{1}{2}\ \Gamma\left(\frac{1}{2}\right)}{S^{5/2}}$$

$$=\frac{3\sqrt{\pi}}{4s^{5/2}} \quad \left[\therefore \Gamma(1/2)=\sqrt{\pi}\right]$$

ii)

$$L\left[e^{-at}\cos bt\right]=\left[L(\cos bt)\right]_{s\to s+a}$$

$$=\left[\frac{s}{s^2+b^2}\right]_{s\to s+a}$$

$$=\left[\frac{s+a}{(s+a)^2+b^2}\right]$$

Problem 3 Find $L\left[\sin 8t\cos 4t+\cos^3 4t+5\right]$

Solution:

$$L\left[\sin 8t\cos 4t+\cos^3 4t+5\right]=L\left[\sin 8t\cos 4t\right]+L\left[\cos^3 4t\right]+L[5]$$

$$L\left[\sin 8t+\cos 4t\right]=L\left[\frac{\sin 12t+\sin 4t}{2}\right] \quad \left[\because \sin A\cos B=\frac{\sin(A+B)+\sin(A-B)}{2}\right]$$

$$= \frac{1}{2}\left\{L[\sin 12t] + L(\sin 4t)\right\}$$

$$= \frac{1}{2}\left\{\frac{12}{s^2+144} + \frac{4}{s^2+16}\right\}$$

$$L\left[\cos^3 4t\right] = L\left[\frac{\cos 12t + 3\cos 4t}{4}\right]\left[\because \cos^3\theta = \frac{\cos 3\theta + 3\cos\theta}{4}\right]$$

$$= \frac{1}{4}\left\{L(\cos 12t) + 3L(\cos 4t)\right\}$$

$$= \frac{1}{4}\left[\frac{s}{s^2+144} + \frac{3s}{s^2+16}\right]$$

$$L[5] = 5L[1] = 5\left[\frac{1}{s}\right] = \frac{5}{s}.$$

$$L\left[\sin 8t\cos 4t + \cos^3 4t + 5\right] = \frac{1}{2}\left\{\frac{12}{s^2+144} + \frac{4}{s^2+16}\right\} + \frac{1}{4}\left\{\frac{s}{s^2+144} + \frac{3s}{s^2+16}\right\} + \frac{5}{s}.$$

Problem 4 Find $L\{f(t)\}$ where $f(t) = \begin{cases} 0 & ; when \quad 0 < t < 2 \\ 3 & ; when \quad t > 2 \end{cases}$.

Solution:

W.K.T $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st} 0.dt + \int_2^\infty e^{-st} 3dt$$

$$= 3\int_2^\infty e^{-st} dt = 3\left[\frac{e^{-st}}{-s}\right]_2^\infty$$

$$= 3\left[\frac{e^{-\infty} - e^{-2s}}{-s}\right] = \frac{3e^{-2s}}{s}.$$

Problem 5 If $L[f(t)] = F(s)$ show that $L\{f(\text{at})\} = \frac{1}{a}F\left(\frac{s}{a}\right)$.

(OR)

State and prove change of scale property.

Solution:

W.K.T $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$L\{f(\text{at})\} = \int_0^\infty e^{-st} f(\text{at})\, dt$$

Put $at = x$ when $t = 0$, $x = 0$
$adt = dx$ when $t = \infty$, $x = \infty$

$$L\{f(\text{at})\} = \int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x) \frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)t} f(t)\, dt \quad [\because x \text{ is a dummy variable}]$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right).$$

Problem 6 Does Laplace transform of $\frac{\cos \text{at}}{t}$ exist? Justify

Solution:

If $L\{f(t)\} = F(s)$ and $\frac{1}{t} f(t)$ has a limit as $t \to 0$ then $L\left\{\frac{f(t)}{t}\right\} = \int_S^\infty F(s)\, ds$.

Here $\lim_{t\to 0} \frac{\cos \text{at}}{t} = \frac{1}{0} = \infty$

$\therefore L\left\{\frac{\cos \text{at}}{t}\right\}$ does not exist.

Problem 7 Using Laplace transform evaluate $\int_0^\infty te^{-3t} \sin 2t\, dt$

Solution:

W.K.T $L\{f(t)\} = \int_0^\infty e^{st} f(t)\, dt$

$$= \int_0^\infty e^{-3t} t \sin 2t\, dt = L[(t \sin 2t)]_{s=3}$$

$$= \left[-\frac{d}{ds} L(\sin 2t)\right]_{s=3} = \left[-\frac{d}{ds}\left(\frac{2}{s^2+4}\right)\right]_{s=3}$$

$$= \left[-\left(\frac{-4s}{\left(s^2+4\right)^2}\right)\right]_{s=3}$$

$$= \left[-\left(\frac{4s}{\left(s^2+4\right)^2}\right)\right]_{s=3} = \frac{12}{169}.$$

Problem 8 Find $L\left[\int_0^t \frac{\sin u}{u}\,du\right]$

Solution:

By Transform of integrals, $L\left[\int_0^t f(x)\,dx\right] = \frac{1}{s} L\{f(t)\}$

$$L\left[\int_0^t \frac{\sin u}{u}\,du\right] = \frac{1}{s} L\left[\frac{\sin t}{t}\right] = \frac{1}{s}\int_s^\infty L[\sin t]\,ds = \frac{1}{s}\int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \frac{1}{s}\left[\tan^{-1} s\right]_s^\infty = \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1} s\right]$$

$$= \frac{1}{s}\cot^{-1} s$$

Problem 9 Find the Laplace transform of the unit step function.

Solution:

The unit step function (Heaviside's) is defined as

$U_a(t) = \begin{cases} 0 & ; \quad t < a \\ 1 & ; \quad t > a \end{cases}$, where $a \geq 0$

W.K.T $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$L\{U_a(t)\} = \int_0^\infty e^{-st} U_a(t)\,dt$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt$$

$$= \int_a^\infty e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \left[\frac{e^{-\infty} - e^{-as}}{-s}\right] = \frac{e^{-as}}{s}$$

Thus $L\{U_a(t)\} = \frac{e^{-as}}{s}$

Problem 10 Find the inverse Laplace transform of $\frac{1}{(s+a)^n}$

Solution:

$$L\left(t^n\right) = \frac{n!}{s^{n+1}}$$

$$L\left(t^{n-1}\right)=\frac{(n-1)!}{s^n}$$

$$L\left(e^{-at}t^{n-1}\right)=\left[\frac{(n-1)!}{s^n}\right]_{s\to s+a}=\frac{(n-1)!}{(s+a)^n}$$

$$e^{-at}t^{n-1}=L^{-1}\left(\frac{(n-1)!}{(s+a)^n}\right)$$

$$e^{-at}t^{n-1}=(n-1)!L^{-1}\left[\frac{1}{(s+a)^n}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s+a)^n}\right]=\frac{1}{(n-1)!}e^{-at}t^{n-1}$$

## Problem 11 Find the inverse Laplace Transform of $\frac{1}{s\left(s^2+a^2\right)}$

## Solution:

W.K.T $L^{-1}\left[\frac{1}{s}F(s)\right]=\int_0^t L^{-1}[F(s)]dt$

$$L^{-1}\left[\frac{1}{s\left(s^2+a^2\right)}\right]=\int_0^t L^{-1}\left[\frac{1}{s^2+a^2}\right]dt$$

$$=\int_0^t \frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right]dt$$

$$=\frac{1}{a}\int_0^t \sin at \; dt$$

$$=\frac{1}{a}\left[\frac{\cos at}{a}\right]_0^t$$

$$=-\frac{1}{a^2}[\cos at-1]$$

$$=\frac{1}{a^2}[1-\cos at].$$

## Problem 12 Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

## Solution:

$$L^{-1}\left[\frac{s}{(s+2)^2}\right]=L^{-1}[sF(s)]=\frac{d}{dt}\quad L^{-1}[F(s)]$$

Where $F(s)=\dfrac{1}{(s+2)^2}$ , $L\left[t^n\right]=\dfrac{n!}{s^{n+1}}$

$$L^{-1}\left[F(s)\right]=L^{-1}\left[\frac{1}{(s+2)^2}\right],\ L(t)=\frac{1}{s^2}$$

$$L^{-1}\left[F(s)\right]=e^{-2t}\,L^{-1}\left[\frac{1}{s^2}\right]=e^{-2t}\,t$$

$$L^{-1}\left[\frac{s}{(s+2)^2}\right]=\frac{d}{dt}\left[e^{-2t}\,t\right]=t\left(-2e^{-2t}\right)+e^{-2t}$$

$$L^{-1}\left[\frac{s}{(s+2)^2}\right]=e^{-2t}\ (1-2t)$$

Problem 13 Find $L^{-1}\left[\dfrac{s+2}{\left(s^2+4s+5\right)^2}\right]$.

Solution:

$$L^{-1}\left[\frac{s+2}{\left(s^2+4s+5\right)^2}\right]=L^{-1}\left[\frac{s+2}{\left((s+2)^2+1\right)^2}\right]$$

$$=e^{-2t}L^{-1}\left[\frac{s}{\left(s^2+1\right)^2}\right].......(1)$$

$$L^{-1}\left[\frac{s}{\left(s^2+1\right)^2}\right]=t\ L^{-1}\int_s^\infty\frac{s}{\left(s^2+1\right)^2}\ ds$$

$$=t\,L^{-1}\int\frac{du}{2u^2}\qquad let\ u=s^2+1,\ du=2sds$$

$$=\frac{t}{2}L^{-1}\left(\frac{-1}{u}\right)$$

$$=\frac{t}{2}L^{-1}\left(\frac{-1}{s^2+1^2}\right)_s^\infty$$

$$=\frac{t}{2}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$=\frac{t}{2}\sin t........(2)$$

Using (2) in (1)

$$L^{-1}\left[\frac{s+2}{\left(s^2+4s+5\right)^2}\right]=e^{-2t}.\frac{t}{2}\sin t=\frac{1}{2}te^{-2t}\ \sin t.$$

Problem 14 Find the inverse Laplace transform of $\frac{100}{s\left(s^2+100\right)}$

Solution:

Consider $\frac{100}{s\left(s^2+100\right)}=\frac{A}{s}+\frac{Bs}{s^2+100}$

$100=A\left(s^2+100\right)+\left(Bs+C\right)(s)$

Put $s=0,\ 100=A(100)$

$A=1$

$s=1,\quad 100=A(101)+B+C$

$B+C=-1$

Equating $s^2$ term

$0=A+B$

$\Rightarrow B=-1$

$\therefore B+C=-1$ i.e., $-1+C=-1$

$C=0$

$$\therefore L^{-1}\left[\frac{100}{s(s^2+100)}\right]=L^{-1}\left[\frac{1}{s}-\frac{s}{s^2+100}\right]$$

$$=L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{s}{s^2+100}\right]$$

$$=1-\cos 10t$$

Problem 15 Solve $\frac{dx}{dy}-2y=\cos 2t$ and $\frac{dy}{dt}+2x=\sin 2t$ given $x(0)=1;\ y(0)=0$

Solution:

$x'-2y=cos2t$

$y'+2x=\sin 2t$ given $x(0)=1;\ y(0)=0$

Taking Laplace Transform we get

$$\left[sL\left(x\right)-x(0)\right]-2L\left[y\right]=L\left[\cos 2t\right]=\frac{s}{s^2+4}$$

$$\therefore sL\left[x\right]-2\,L\left[y\right]=\frac{s}{s^2+4}+1..........(1)$$

$$\left[sL\left(y\right)-y(0)\right]+2L\left[x\right]=L\left[\sin 2t\right]=\frac{2}{s^2+4}$$

$$2L\left[x\right]+sL\left[y\right]=\frac{2}{s^2+4}........(2)$$

$(1)\times 2-s\times(2)$ gives,

$$-\left(s^2+4\right)L\left(y\right)=\frac{-2}{s^2+4}$$

$$\therefore y = -L^{-1}\left[\frac{2}{s^2+4}\right]$$
$$= -\sin 2t$$
$$2x = \sin 2t - \frac{dy}{dt}$$
$$= \sin 2t + 2\cos 2t$$
$$\therefore x = \cos 2t + \frac{1}{2}\sin 2t$$

## Part-B

Problem 1 Find the Laplace transform of $e^{-t}\int_0^t \frac{\sin t}{t}\,dt$

Solution:

$$L\left(\int_0^t \frac{\sin t}{t}\,dt\right) = \frac{1}{s} L\left(\frac{\sin t}{t}\right)$$
$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty L(\sin t)\,ds$$
$$= \int_s^\infty \frac{1}{s^2+1}\,ds$$
$$= \left(\tan^{-1}(s)\right)_s^\infty$$
$$= \frac{\pi}{2} - \tan^{-1}(s)$$
$$= \cot^{-1}(s)$$
$$\therefore L\left(\int_0^t \frac{\sin t}{t}\,dt\right) = \frac{1}{s}\cot^{-1}(s)$$
$$L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right\} = \left[\frac{1}{s}\cot^{-1}(s)\right]_{s\to s+1}$$
$$= \frac{\cot^{-1}(s+1)}{s+1}.$$

Problem 2 Find $\int_0^\infty te^{-2t}\sin 3t\;dt$ using Laplace transforms.

Solution: $L(\sin 3t) = \frac{3}{s^2+9}$

$$L[t\sin 3t] = -\frac{d}{ds}\left(\frac{1}{s^2+9}\right) = \frac{6s}{\left(s^2+9\right)^2}$$

$$\int_0^\infty e^{-st}\left(t\sin 3t\right)dt = L\left[t\sin 3t\right] \quad \left(\text{ by definition}\right)$$

$$= \frac{6s}{\left(s^2+9\right)^2}$$

i.e., $\int_0^\infty te^{-st}\sin 3t\,dt = \frac{6s}{\left(s^2+9\right)^2}$

Putting $s = 2$ we get $\int_0^\infty te^{-2t}\sin 3t\,dt = \frac{12}{169}$

Problem 3 Find the Laplace transform of $t\int_0^t te^{-4t}\cos 3t\,dt + \frac{\sin 5t}{t}dt$

Solution:

$$L\left[t\int_0^t e^{-4t}\cos 3t\,dt\right] = -\frac{d}{ds}L\left[\int_0^t e^{-4t}\cos 3t\,dt\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\left[L\left(e^{-4t}\cos 3t\right)\right]\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\left[L\left(\cos 3t\right)\right]_{s\to s+4}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\left(\frac{s}{s^2+9}\right)_{s\to s+4}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\frac{s+4}{(s+4)^2+9}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s}\frac{\left(s+4\right)}{(s^2+8s+25)}\right]$$

$$= -\frac{d}{ds}\left[\frac{s+4}{s^3+8s^2+25s}\right]$$

$$= -\left[\frac{\left(s^3+8s^2+25s\right)(1)-\left(s+4\right)\left(3s^3+16s+25\right)}{\left(s^3+8s^2+25s\right)^2}\right]$$

$$= -\left[\frac{s^3+8s^2+25s-3s^3+16s^2-25s-12s^2-64s-100}{\left(s^3+8s^2+25s\right)^2}\right]$$

$$= -\left[\frac{-2s^3-20s^2-64s-100}{\left(s^3+8s^2+25s\right)^2}\right]$$

$$= 2\left[\frac{s^3+10s^2+32s+50}{\left(s^3+8s^2+25s\right)^2}\right]$$

$$L\left[\frac{\sin 5t}{t}\right] = \int_s^\infty L(\sin 5t)\,ds$$

$$= \int_s^\infty \frac{5}{s^2+25}\,ds$$

$$= \left[5.\frac{1}{5}\tan^{-1}\left(\frac{s}{5}\right)\right]_s^\infty$$

$$= \left[\tan^{-1}\left(\frac{s}{5}\right)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{5}\right)$$

$$= co\,\mathrm{t}^{-1}\left(\frac{s}{5}\right)$$

$$\therefore L\left[t\int_0^t e^{-4t}\cos 3t\,dt + \frac{\sin 5t}{t}\right] = L\left[t\int_0^t e^{-4t}\cos 3t\,dt\right] + L\left[\frac{\sin 5t}{t}\right]$$

$$= \frac{2\left(s^3+10s^2+32s+50\right)}{\left(s^3+8s^2+25s\right)^2} + \cot^{-1}\left(\frac{s}{5}\right)$$

Problem 4 Find $L\left[t^2e^{2t}\cos 2t\right]$.

Solution:

$$L\left[t^2e^{2t}\cos 2t\right] = (-1)^2\frac{d^2}{ds^2}L\left[e^{2t}\cos^{2t}\right]$$

$$= \frac{d^2}{ds^2}\left[\left(\frac{s}{s^2+4}\right)_{s\to s-2}\right]$$

$$= \frac{d^2}{ds^2}\left[\left(\frac{s-2}{(s-2)^2+4}\right)\right]$$

$$= \frac{d^2}{ds^2}\left[\frac{s-2}{s^2-4s+8}\right]$$

$$= \frac{d}{ds}\left[\frac{\left(s^2-4s+8\right)(1)-(s-2)(2s-4)}{\left(s^2-4s+8\right)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2-4s+8-2s^2+4s+4s-8}{\left(s^2-4s+8\right)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-s^2+4s}{\left(s^2-4s+8\right)^2}\right]$$

$$= \frac{\left(s^2-4s+8\right)\left(-2s+4\right)-\left(s^2+4s\right)2\left(s^2-4s+8\right)\left(2s-4\right)}{\left(s^2-4s+8\right)^4}$$

$$= \frac{\left(s^2-4s+8\right)\left(-2s+4\right)-\left(s^2+4s\right)\left(2s-4\right)}{\left(s^2-4s+8\right)^3}$$

$$= \frac{-2s^3+8s^2-16s+4s^2-16s+32+4s^3-8s^2-16s^2+32s}{\left(s^2-4s+8\right)^3}$$

$$= \frac{2s^3-12s^2+32}{\left(s^2-4s+8\right)^3}.$$

Problem 5 Verify the initial and final value theorems for the function $f(t)=1+e^{-t}(\sin t+\cos t)$

Solution: Given $f(t)=1+e^{-t}(\sin t+\cos t)$

$$L\{f(t)\} = L\{1+e^{-t}\sin t+e^{-t}\cos t\}$$

$$= L(1)+L\left(e^{-t}\sin t\right)+L\left(e^{-t}\cos t\right)$$

$$= \frac{1}{s}+\frac{1}{(s+1)^2+1}+\frac{s+1}{(s+1)^2+1}$$

$$= \frac{1}{s}+\frac{s+2}{(s+1)^2+1}$$

Initial value theorem:

$$\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, sF(s)$$

$$LHS = \underset{t\to 0}{Lt}\left[1+e^{-t}(\sin t+\cos t)\right] = 1+1 = 2$$

$$RHS = \underset{s\to\infty}{Lt}\, sF(s)$$

$$= \underset{s\to\infty}{Lt}\, s\left[\frac{1}{s}+\frac{s+2}{(s+1)^2+1}\right]$$

$$= \underset{s\to\infty}{Lt}\left[1+\frac{s^2+2s}{s^2+2s+2}\right]$$

$$= \underset{s\to\infty}{Lt} \left[ \frac{2s^2+4s+2}{s^2+2s+2} \right]$$

$$= \underset{s\to\infty}{Lt} \left[ \frac{2+4/s+2/s^2}{1+2/s+2/s^2} \right] = 2$$

$LHS = RHS$

Hence initial value theorem is verified.

Final value theorem:

$$\underset{t\to\infty}{Lt}\, f(t) = \underset{s\to 0}{Lt}\, sF(s)$$

$$LHS = \underset{t\to\infty}{Lt}\, f(t)$$

$$= \underset{t\to\infty}{Lt} \left(1+e^{-t}\sin t + e^{-t}\cos t\right) = 1 \quad \left(\because e^{-\infty}=0\right)$$

$$RHS = \underset{s\to 0}{Lt}\, sF(s)$$

$$= \underset{s\to 0}{Lt}\, s\left[ \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} \right]$$

$$= \underset{s\to 0}{Lt} \left[ 1 + \frac{s^2+2s}{(s+1)^2+1} \right] = 1$$

$LHS = RHS$

Hence final value theorem is verified.

Problem 6 Find $L^{-1}\left[\log\left(\frac{s^2+1}{s^2}\right)\right]$.

Solution:

$$L^{-1}\left[F(s)\right] = -\frac{1}{t} L^{-1}\left[F'(s)\right] \text{.......(1)}$$

$$F(s) = \log\left(\frac{s^2+1}{s^2}\right)$$

$$F'(s) = \frac{d}{ds}\log\left[\left(s^2+1\right) - \log\left(s^2\right)\right]$$

$$= \frac{2s}{s^2+1} - \frac{2s}{s^2}$$

$$L^{-1}\left[F'(s)\right] = L^{-1}\left[\frac{2s}{s^2+1} - \frac{2s}{s^2}\right]$$

$$= 2L^{-1}\left[\frac{s}{s^2+1} - \frac{1}{s}\right]$$

$$= 2\left[\cos t - 1\right]$$

$$L^{-1}\left[\log\left(\frac{s^2+1}{s^2}\right)\right] = -\frac{1}{t}2\left[\cos t - 1\right]$$

$$= \frac{2(1-\cos t)}{t}.$$

Problem 7 Find the inverse Laplace transform of $\frac{s+3}{(s+1)(s^2+2s+3)}$

Solution: $\frac{s+3}{(s+1)(s^2+2s+3)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+3}$ ....(1)

$$s+3 = A(s^2+2s+3) + (Bs+c)(s+1)$$

Put $s=-1$

2=2A

A = 1

Equating the coefficients of $s^2$

0 = A+B $\Rightarrow$ B = -1

Put s = 0

3 = 3A+C

C = 0

$$(1) \Rightarrow \frac{s+3}{(s+1)(s^2+2s+3)} = \frac{1}{s+1} - \frac{s}{s^2+2s+3}$$

$$= \frac{1}{s+1} - \frac{s}{(s+1)^2+2}$$

$$= \frac{1}{s+1} - \frac{s+1}{(s+1)^2+2} + \frac{1}{(s+1)^2+2}$$

$$L^{-1}\left[\frac{s+3}{(s+1)(s^2+2s+3)}\right] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{s+1}{(s+1)^2+2}\right] + L^{-1}\left[\frac{1}{(s+1)^2+2}\right]$$

$$= e^{-t} - e^{-t}L^{-1}\left[\frac{s}{s^2+2}\right] + e^{-t}L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= e^{-t} - e^{-t}\cos\sqrt{2}t + e^{-t}\sin\sqrt{2}t$$

$$= e^{-t}\left[1-\cos\sqrt{2}t + \sin\sqrt{2}t\right].$$

Problem 8 Find $L^{-1}\left[s\log\left(\frac{s-1}{s+1}\right)+2\right]$

Solution:

$$L^{-1}\left[s\log\left(\frac{s-1}{s+1}\right)+2\right] = f(t)$$

$$\therefore L[f(t)] = s\log\left(\frac{s-1}{s+1}\right)+2$$

$$= s\log(s-1) - s\log(s+1) + 2$$

$$L\{t\,f(t)\} = -\frac{d}{ds}\left[s\log(s-1) - s\log(s+1) + 2\right]$$

$$= -\left[\frac{s}{s-1} + \log(s-1) - \frac{s}{s+1} - \log(s+1)\right]$$

$$= -\left[\log\left(\frac{s-1}{s+1}\right) + \frac{s(s+1) - s(s-1)}{s^2-1}\right]$$

$$= \log\left(\frac{s-1}{s+1}\right) - \left(\frac{s^2+s-s^2+s}{s^2-1}\right)$$

$$= \log\left(\frac{s+1}{s-1}\right) - \frac{2s}{s^2-1}$$

$$tf(t) = L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] - 2L^{-1}\left[\frac{s}{s^2-1}\right]$$

$$= L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] - 2\cosh t....(1)$$

To find $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

Let $f(t) = L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

$$L\left[f(t)\right] = \log\left(\frac{s+1}{s-1}\right)$$

$$L\{t\,f(t)\} = -\frac{d}{ds}\left[\log(s+1) - \log(s-1)\right]$$

$$= \frac{1}{s-1} - \frac{1}{s+1} = \frac{2}{s^2-1}$$

$$\therefore t\;f(t) = 2L^{-1}\left[\frac{1}{s^2-1}\right] = 2\sinh t$$

$$f(t) = \frac{2\sinh t}{t}.....(2)$$

Using (2) in (1)

$$tf(t) = \frac{2\sinh t}{t} - 2\cosh t$$

$$f(t) = \frac{2\sinh t}{t^2} - \frac{2\cosh t}{t}$$

$$= 2\left[\frac{\sinh t - t\cosh t}{t^2}\right].$$

Problem 9 Using convolution theorem find $L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]$

Solution:

$$L^{-1}\left[F(s)G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$=L^{-1}\left[\frac{s}{s^2+a^2}\right]*\frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$=\cos \text{at}*\frac{1}{a}\sin \text{at}$$

$$=\frac{1}{a}\left[\cos \text{at}*\sin \text{at}\right]$$

$$=\frac{1}{a}\int_0^t \cos \text{a}\,u \sin a\left(t-u\right)du$$

$$=\frac{1}{a}\int_0^t \sin\left(\text{at}-au\right)\cos \text{a}\,u\,du$$

$$=\frac{1}{a}\int_0^t \frac{\sin\left(\text{at}-au+au\right)+\sin\left(\text{at}-au-au\right)}{2}du$$

$$=\frac{1}{2a}\int_0^t\left[\sin \text{at}+\sin a\left(t-2u\right)\right]du$$

$$=\frac{1}{2a}\left[\left(\sin \text{at}\right)u-\frac{\cos a\left(t-2u\right)}{-2a}\right]_0^t$$

$$=\frac{1}{2a}\left[t\sin \text{at}+\frac{\cos \text{at}}{2a}-\frac{\cos \text{at}}{2a}\right]$$

$$=\frac{t\sin \text{at}}{2a}.$$

Problem 10 Find the Laplace inverse of $\frac{1}{(s+1)\left(s^2+9\right)}$ using convolution theorem.

Solution:

$$L^{-1}\left[F(s).G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{1}{(s+1)\left(s^2+9\right)}\right]=L^{-1}\left[\frac{1}{(s+1)}.\frac{1}{\left(s^2+9\right)}\right]$$

$$= L^{-1}\left[\frac{1}{(s+1)}\right] * L^{-1}\left[\frac{1}{(s^2+9)}\right]$$

$$= e^t * \frac{1}{3}\sin 3t$$

$$= \frac{1}{3}\int_0^t e^{-u}\sin\left[3(t-u)\right]du$$

$$= \frac{1}{3}\int_0^t e^{-u}\sin(3t-3u)\,du$$

$$= \frac{1}{3}\int_0^t e^{-u}\left[\sin 3t\cos 3u - \cos 3t\sin 3u\right]du$$

$$= \frac{1}{3}\sin 3t\int_0^t e^{-u}\cos 3u\;du - \frac{1}{3}\cos 3t\int_0^t e^{-u}\sin 3u\;du$$

$$= \frac{\sin 3t}{3}\left[\frac{e^{-u}}{10}(-\cos 3u + 3\sin 3u)\right]_0^t - \frac{\cos 3t}{3}\left[\frac{e^{-u}}{10}(-\sin 3u - 3\cos 3u)\right]_0^t$$

$$= \frac{\sin 3t}{3}\left[\frac{e^{-u}}{10}(-\cos 3t + 3\sin 3t) - \frac{1}{10}(-1)\right]$$

$$= \frac{\sin 3t}{3}\left[\frac{e^{-u}}{10}(-\sin 3t - 3\cos 3t) - \frac{1}{10}(-3)\right]$$

Problem 11 Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ using convolution theorem

Solution:

$$L^{-1}\left[F(s).G(s)\right] = L^{-1}\left[F(s)\right] * L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \frac{1}{2}\left[\frac{\sin\left[(a-b)u+bt\right]}{a-b} + \frac{\sin\left[(a+b)u-bt\right]}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(at-bt+bt)}{a-b} + \frac{\sin(at+bt-bt)}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at}{a^2-b^2} - \frac{2b\sin bt}{a^2-b^2}\right]$$

$$=\frac{1}{2}\left[\frac{2a\sin at-2b\sin bt}{a^2-b^2}\right]$$
$$=\frac{a\sin at-b\sin bt}{a^2-b^2}.$$

Problem 12 Using convolution theorem find the inverse Laplace transform of $\frac{1}{\left(s^2+a^2\right)^2}$.

Solution:

$$L^{-1}\left[F(s).G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$
$$L^{-1}\left[\frac{1}{\left(s^2+a^2\right)^2}\right]=L^{-1}\left[\frac{1}{s^2+a^2}\right]*L^{-1}\left[\frac{1}{s^2+a^2}\right]$$
$$=\frac{\sin \text{at}}{a}*\frac{\sin \text{at}}{a}$$
$$=\frac{1}{a^2}\int_0^t \sin au\sin a(t-u)\,du$$
$$=\frac{1}{2a^2}\int_0^t\left[\cos(2au-\text{at})-\cos\text{at}\right]du\quad\left[\because 2\sin A\sin B=\cos(A-B)-\cos(A+B)\right]$$
$$=\frac{1}{2a^2}\left[\frac{\sin(2au-\text{at})}{2a}-(\cos\text{at})u\right]_0^t$$
$$=\frac{1}{2a^2}\left[\frac{\sin\text{at}}{2a}-t\cos\text{at}-\left(\frac{-\sin\text{at}}{2a}\right)\right]$$
$$=\frac{1}{2a^2}\left[\frac{2\sin\text{at}}{2a}-t\cos\text{at}\right]$$
$$=\frac{1}{2a^3}\left[\sin\text{at}-at\cos\text{at}\right]$$

Problem 13 Solve the equation $y''+9y=\cos 2t;\ y(0)=1\text{and } y(\pi/2)=-1$

Solution:

Given $y''+9y=\cos 2t$

$$L\left[y''(t)+9y(t)\right]=L\left[\cos 2t\right]$$
$$L\left[y''(t)\right]+9L\left[y(t)\right]=L\left[\cos 2t\right]$$
$$\left[s^2L\left[y(t)\right]-sy(0)-y'(0)\right]+9L\left[y(t)\right]=\frac{s}{s^2+4}$$

As $y'(0)$ is not given, it will be assumed as a constant, which will be evaluated at the end. $\therefore y'(0) = A.$

$$L\left[y(t)\right]\left[s^2+9\right]-s-A=\frac{s}{s^2+4}$$

$$L\left[y(t)\right]\left[s^2+9\right]=\frac{s}{s^2+4}+s+A$$

$$L\left[y(t)\right]=\frac{s}{\left(s^2+4\right)\left(s^2+9\right)}+\frac{s}{s^2+9}+\frac{A}{s^2+9}.$$

Consider $\frac{s}{\left(s^2+4\right)\left(s^2+9\right)}=\frac{As+B}{s^2+4}+\frac{Cs+D}{s^2+9}$

$$s=(As+B)\left(s^2+9\right)+(Cs+B)\left(s^2+4\right)$$

$$= As^3+9As+Bs^2+9B+Cs^3+4(s+1)s^2+4$$

Equating coefficient of $s^3$ A + C = 0 …….(1)
Equating coefficient of $s^2$ B + D = 0 …….(2)
Equating coefficient of $s$ 9A+ 4C = 1 …….(3)
Equating coefficient of constant 9B +4D = 0 …….(4)
Solving (1) & (3)

$$4A+4C=0$$
$$-9A+4C=-1$$
$$-5A=-1$$
$$A=\frac{1}{5}$$
$$\frac{1}{5}+C=0$$
$$C=-\frac{1}{5}$$

Solving (2) & (4)

$$9B+9D=0$$
$$9B+AD=0$$
$$D=0$$
$$\therefore B=0\,\&\,D=0.$$

$$\therefore \frac{s}{\left(s^2+4\right)\left(s^2+9\right)}=\frac{1}{5}\frac{s}{s^2+4}-\frac{s}{5\left(s^2+9\right)}$$

$$\therefore L\left[y(t)\right]=\frac{1}{5}\left\{\frac{s}{s^2+4}-\frac{s}{s^2+9}\right\}+\frac{s}{s^2+9}+\frac{A}{s^2+4}$$

$$\therefore y(t)=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{A}{3}\sin 3t$$

$$=\frac{1}{5}\cos 2t+\frac{4}{5}\cos 3t+\frac{A}{3}\sin 3t$$

Given $y\left(\frac{\pi}{2}\right)=-1$

$-1=-\frac{1}{5}-\frac{A}{5}$

$\therefore A=\frac{12}{5}$

$\therefore y(t)=\frac{1}{5}\cos 2t+\frac{4}{5}\cos 3t+\frac{4}{5}\sin 3t$

Problem 14 Using Laplace transform solve $\frac{d^2y}{dx^2}-\frac{3dy}{dx}+2y=4$ given that $y(0)=2,\ y'(0)=3$

Solution:

$L\left[y''(t)\right]-3L\left[y'(t)\right]+2L\left[y(t)\right]=L[4]$

$s^2L\left[y(t)\right]-sy(0)-y'(0)-3sL\left[y(t)\right]+3y(0)+2L\left[y(t)\right]=\frac{4}{5}$

$\left(s^2-3s+2\right)L\left[y(t)\right]-2s-3+6=\frac{4}{s}$

$\left(s^2-3s+2\right)L\left[y(t)\right]=\frac{4}{s}+2s-3$

$L\left[f(t)\right]=\frac{2s^2-3s+4}{s\left(s^2-3s+2\right)}$

$$L\left[f(t)\right]=\frac{2s^2-3s+4}{s(s-1)(s-2)}$$

$\frac{2s^2-3s+4}{s(s-1)(s-2)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-2}$

$2s^2-3s+4=A(s-1)(s-2)+Bs(s-2)+Cs(s-1)$

Put s = 0 4 = 2A => A = 2

s = 1 $3=-B \Rightarrow B=-3$

$s=2$ $6=2c \Rightarrow C=3$

$\therefore L\left[y(t)\right]=\frac{2}{s}-\frac{3}{s-1}+\frac{3}{s-2}$

$y(t)=2-3e^t+3e^{2t}$

Problem 15 Solve $\frac{dx}{dy}+y=\sin t;\ x+\frac{dy}{dt}\cos t$ with $x=2$ and $y=0$ when $t=0$

Solution:

Given $x'(t)+y(t)=\sin t$

$x(t)+y'(t)=\cos t$

$L\left[x'(t)\right]+L\left[y(t)\right]=L[\sin t]$

$sL\left[x'(t)\right]-x(0)+L\left[y(t)\right]=\dfrac{1}{s^2+1}$

$sL\left[x(t)\right]+L\left[y(t)\right]=\dfrac{1}{s^2+1}+2.........(1)$

$L\left[x(t)\right]+L\left[y'(t)\right]=L[\cos 2t]$

$L\left[x(t)\right]+sL\left[y(t)\right]-=y(0)=\dfrac{1}{s^2+1}....(2)$

Solving (1) & (2)

$\left(1-s^2\right)L\left[y(t)\right]=2+\dfrac{1-s^2}{s^2+1}$

$\left(1-s^2\right)L\left[y(t)\right]=\dfrac{2s^2+2+1-s^2}{s^2+1}$

$L\left[y(t)\right]=\dfrac{2s^2+3}{\left(s^2+1\right)\left(1-s^2\right)}$

$\dfrac{s^2+3}{\left(s^2+1\right)\left(1-s^2\right)}=\dfrac{As+B}{s^2+1}+\dfrac{Cs+D}{1-s^2}$

$s^2+3=(As+B)\left(1-s^2\right)+(Cs+D)\left(s^2+1\right)$

Equating $s^3$ on both sides

$0=-A+C$ $\qquad$ $put\, s=0$

$A=c$ $\qquad$ $3=B+D$

$A=0 \;\; C=0$

Equating $s^2$ on both sides

$1=-B+D$ $\qquad$ $D=2$

$B=1$

Equation son both sides 0 = A + B

$\Rightarrow y(t)=L^{-1}\left[\dfrac{1}{s^2+1}\right]-2L^{-1}\left[\dfrac{1}{s^2+1}\right]$

$\quad =\sin t-2\sinh t$

To find $x(t)$ we have $x(t)+y'(t)=\cos t$, $x(t)=\cos t-y'(t)$, $y(t)=\sin t-2\sinh t$

$\dfrac{dy}{dt}=\cos t-2\cosh t$

$x(t)=\cos t-\cos t+2\cosh t$

$\quad =2\cosh t$

Hence $x(t)=2\cosh t$

$y(t)=\sin t-2\sinh t$