Stoke's theorem:

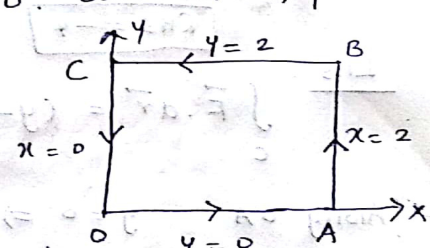
If $\vec{F}$ is any continuous differentiable vector function and S is a surface enclosed by a curve C, then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot \hat{n}\, ds$$

(1) Verify stoke's theorem for $\vec{F} = (y-z+2)\vec{i} + (yz+4)\vec{j} - xz\vec{k}$ where S is the open surface of the cube $x=0, y=0, z=0, x=2, y=2, z=2$ above the XY plane.

Soln:

Above the XY plane, $z=0$. Consider XY plane.



By Stoke's theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\hat{n}\, ds$$

RHS:

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-xz) - \frac{\partial}{\partial z}(yz+4)\right] - \vec{j}\left[\frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial z}(y-z+2)\right] + \vec{k}\left[\frac{\partial}{\partial x}(yz+4) - \frac{\partial}{\partial y}(y-z+2)\right]$$

$$= -y\vec{i} + (z-1)\vec{j} - \vec{k}$$

Faces	$\hat{n}$	ds	Limit	$curl\,\vec{F}\cdot\hat{n}$	$\iint curl\,\vec{F}\cdot\hat{n}\,ds$
$S_1(x=2)$	$\vec{i}$	$dy\,dz$	$y:0\to 2$ $z:0$ to 2	$-y$	$I_1=\int_0^2\int_0^2 -y\,dy\,dz=-4$
$S_2(x=0)$	$-\vec{i}$	$dy\,dz$	$y:0$ to 2 $z:0$ to 2	y	$I_2=\int_0^2\int_0^2 y\,dy\,dz=4$
$S_3(y=2)$	$\vec{j}$	$dx\,dz$	$x:0$ to 2 $z:0$ to 2	$z-1$	$I_3=\int_0^2\int_0^2 (z-1)\,dx\,dz=0$
$S_4(y=0)$	$-\vec{j}$	$dx\,dz$	$x:0$ to 2 $z:0$ to 2	$-z+1$	$I_4=\int_0^2\int_0^2 (-z+1)\,dx\,dz=0$
$S_5(z=2)$	$\vec{k}$	$dy\,dx$	$x:0$ to 2 $y:0$ to 2	-1	$I_5=\int_0^2\int_0^2 (-1)\,dx\,dy=-4$

$$RHS = I_1+I_2+I_3+I_4+I_5 = -4+4-4 = -4$$

$$\boxed{RHS=-4}$$

LHS :

$$\int_C \vec{F}\cdot d\vec{r} = (y-z+2)\,dx+(yz+4)\,dy-xz\,dz$$

Along OA: $y=0 \Rightarrow dy=0$, $z=0 \Rightarrow dz=0$, $x:0$ to 2

$$I_1=\int_0^2 2\,dx=(2x)_0^2=4$$

Along AB: $x=2,\ z=0$, $dx=0,\ dz=0$, $y:0$ to 2

$$I_2=\int_0^2 4\,dy=8$$

Along BC: $y=2,\ z=0$, $dy=0,\ dz=0$, $x:2$ to 0

$$I_3=\int_2^0 4\,dx=-8$$

Along CO: $x=0,\ z=0$, $dx=0,\ dz=0$, $y:2$ to 0

$$I_4=\int_2^0 4\,dy=-8$$

$$LHS = I_1+I_2+I_3+I_4 = 4+8-8-8$$

$$\boxed{LHS=-4} \Rightarrow \boxed{LHS=RHS}$$