Fourier Series :

If $f(x)$ is a periodic function, it satisfies the Dirichlet's condition then it can be represented by an infinite series called as Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where,

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

This is Fourier series of $f(x)$ defined in the interval $c \le x \le c+2l$. Then a_0, a_n and b_n are called Fourier constant (or) Euler's formula.

FORMULA:

Limit	$f(x)$	Fourier constant
$(0,2l)$	$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$	$a_0=\frac{1}{l}\int_0^{2l}f(x)\,dx$ $a_n=\frac{1}{l}\int_0^{2l}f(x)\cos\frac{n\pi x}{l}\,dx$ $b_n=\frac{1}{l}\int_0^{2l}f(x)\sin\frac{n\pi x}{l}\,dx$
$(-l,l)$	$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$	$a_0=\frac{1}{l}\int_{-l}^{l}f(x)\,dx$ $a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\,dx$ $b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}\,dx$

Problems based on the interval $(0,2l)$:

1) Find the Fourier Series for the function $f(x)=x^2$ in the interval $(0,2l)$

Soln:

Fourier series:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right) \longrightarrow (1)$$

To find a_0 :

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{l}\int_0^{2l} x^2\,dx$$

$$= \frac{1}{l}\left[\frac{x^3}{3}\right]_0^{2l}$$

$$\boxed{a_0 = \frac{8l^2}{3}}$$

To find a_n :

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^{2l} x^2\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\frac{l}{n\pi}x^2\sin\frac{n\pi x}{l} + 2x\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} + 2\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)^3\right]_0^{2l}$$

$u = x^2$, $u' = 2x$, $u'' = 2$, $u''' = 0$

$v = \cos\frac{n\pi x}{l}$, $v_1 = \frac{\sin\frac{n\pi x}{l}}{n\pi/l}$, $v_2 = \frac{-\cos\frac{n\pi x}{l}}{(n\pi/l)^2}$, $v_3 = \frac{-\sin\frac{n\pi x}{l}}{(n\pi/l)^3}$

$$= \frac{1}{l}\left[\frac{2l^3}{n^2\pi^2}(1) + \frac{2l^3}{n^2\pi^2}\cos 0\right]$$

$$\boxed{a_n = \frac{4l^2}{n^2\pi^2}}$$

To find b_n:

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^{2l} x^2 \sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[-x^2\frac{l}{n\pi}\cos\frac{n\pi x}{l} + 2x\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right) + \frac{2l^3}{n^3\pi^3}\cos\left(\frac{n\pi x}{l}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[-\frac{4l^3}{n\pi} + \frac{2l^3}{n^3\pi^3} - \frac{2l^3}{n^3\pi^3}\right]$$

$$\boxed{b_n = -\frac{4l^2}{n\pi}}$$

Subs a_0, a_n and b_n in (1),

$$f(x) = \frac{4l^2}{3} + \sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} - \frac{4l^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}.$$

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(2) Find the Fourier Series for $f(x) = 2x - x^2$ in $0 < x < 2$.

Soln: Here $2l = 2 \Rightarrow \boxed{l = 1}$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \longrightarrow (1)$$

To find a_0:

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{1}\int_0^{2} f(x)\,dx \quad (\because l = 1)$$

$$= \int_0^{2} (2x - x^2)\,dx$$

$$= \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{12-8}{3}$$

$$\boxed{a_0 = \frac{4}{3}}$$

To find a_n:

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{1}\int_0^{2} f(x)\cos n\pi x\,dx$$

$$= \int_0^{2} (2x - x^2)\cos n\pi x\,dx$$

$$= \left\{(2x - x^2)\frac{\sin n\pi x}{n\pi} + \frac{2-2x}{n^2\pi^2}\cos n\pi x + 2\frac{\sin n\pi x}{n^3\pi^3}\right\}_0^2$$

$$= \left[\frac{-2}{n^2\pi^2} - \frac{2}{n^2\pi^2}\right].$$

$$\boxed{a_n = \frac{-4}{n^2\pi^2}}$$

To find b_n:

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{1}\int_0^{2} f(x)\sin n\pi x\,dx$$

$$= \left[-(2x-x^2)\frac{\cos n\pi x}{n\pi} + (2-2x)\frac{\sin n\pi}{n^2\pi^2} - 2\frac{\cos n\pi x}{n^3\pi^3}\right]_0^2$$

$$\boxed{b_n = 0.}$$

Subs a_0, a_n and b_n in (1),

$$f(x) = \frac{2}{3} + \sum_{n=1}^{\infty}\left(\frac{-4}{n^2\pi^2}\right)\cos n\pi x$$

(3) Find the Fourier series for

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \end{cases}$$

Soln:

Here $2l = 2 \Rightarrow l = 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \longrightarrow (1)$$

To find a_0:

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{1}\int_0^{2} f(x)\,dx$$

$$a_0 = \int_0^1 x\,dx + \int_1^2 (2-x)\,dx$$

$$= \left(\frac{x^2}{2}\right)_0^1 + \left(2x - \frac{x^2}{2}\right)_1^2$$

$$\boxed{a_0 = 1}$$

To find a_n :

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \int_0^1 x\cos n\pi x\,dx + \int_1^2 (2-x)\cos n\pi x\,dx$$

$$= \left[x\frac{\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]_0^1 + \left[(2-x)\frac{\sin n\pi x}{n\pi} - \frac{\cos n\pi x}{n^2\pi^2}\right]_1^2$$

$$= \frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} + \frac{(-1)^n}{n^2\pi^2}$$

$$\boxed{a_n = \frac{2}{n^2\pi^2}[(-1)^n - 1)}$$

To find b_n :

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{1}\int_0^2 f(x)\sin\frac{n\pi x}{1}\,dx$$

$$= \int_0^1 x\sin\frac{n\pi x}{1}\,dx + \int_1^2 (2-x)\sin n\pi x\,dx$$

$$= \left[-x \frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2} \right]_0^1 + \left[\frac{-(2-x)\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2} \right]_1^2$$

$$\boxed{b_n = 0}$$

Subs a_0, a_n and b_n in (1),

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right] \cos n\pi x$$

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(4) Find the Fourier Series for the function $f(x) = x^2$ in $(0, 2\pi)$

Soln:

Here $2l = 2\pi$

$$\boxed{l = \pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\pi} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\pi}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \rightarrow (1)$$

To find a_0:

$$a_0 = \frac{1}{l} \int_0^{2l} f(x)\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x^2\, dx$$

$$= \frac{1}{\pi} \left[\frac{x^3}{3} \right]_0^{2\pi} = \frac{8\pi^3}{3\pi}$$

$$\boxed{a_0 = \frac{8\pi^2}{3}}$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx \, dx$$

$$= \frac{1}{\pi}\left[x^2 \frac{\sin nx}{n} + \frac{2x\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{4\pi}{n^2} - 0\right]$$

$$\boxed{a_n = \frac{4}{n^2}}$$

To find b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx \, dx$$

$$= \frac{1}{\pi}\left[-x^2 \frac{\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right]$$

$$\boxed{b_n = \frac{-4\pi}{n}}$$

Subs a_0, a_n and b_n in ①,

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx - \sum_{n=1}^{\infty} \frac{4\pi}{n}\sin nx$$