PARSEVAL'S IDENTITY :

If the Fourier series corresponding to $f(x)$ converges uniformly to $f(x)$ in $(-l, l)$ then,

$$\frac{1}{l}\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

For the interval $(0, 2l)$, Parseval's identity is,

$$\frac{1}{l}\int_{0}^{2l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Parseval's identity for half range cosine series is,

$$\frac{2}{l}\int_{0}^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}a_n^2$$

Parseval's identity for half range sine series is,

$$\frac{2}{l}\int_{0}^{l}[f(x)]^2\,dx = \sum_{n=1}^{\infty}b_n^2$$

Deduction :

* Find the Fourier Series
* Expand the Fourier Series

Problems:

(1) Find the Sum of the Fourier Series for

$$f(x) = \begin{cases} x^2, & -\pi < x \le 0 \\ 0, & 0 \le x < \pi \end{cases}$$

(i) at $x = \frac{\pi}{2}$ (ii) at $x = -\frac{\pi}{2}$

(iii) at $x = -\pi$ (iv) at $x = \pi$ (v) at $x = 0$.

Soln:

(i) $x = \frac{\pi}{2}$ is a continuous point in $(0, \pi)$

$$f(x) = 0$$

(ii) $x = -\frac{\pi}{2}$ is a continuous point in $(-\pi, 0)$.

$$f(x) = f\left(-\frac{\pi}{2}\right) = \left(-\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}$$

(iii) $x = -\pi$ is a discontinuous point in the end.

$$\therefore f(x) = \frac{f(-\pi) + f(\pi)}{2} = \frac{(-\pi)^2 + 0}{2} = \frac{\pi^2}{2}$$

(iv) $x = \pi$ is a discontinuous point in the end

$$f(x) = \frac{f(-\pi) + f(\pi)}{2} = \frac{(-\pi)^2 + 0}{2} = \frac{\pi^2}{2}$$

(v) $x = 0$ is a Continuous point.

$$f(x) = f(0) = 0.$$

(2) Find the Sum of the Fourier series for

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2, & 1 < x < 2 \end{cases}$$

(i) at $x = 1$. (ii) at $x = 0$

Soln:

(i) $x = 1$ is a discontinuous point in the middle.

$$f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} 1-h = 1$$

$$f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} 2 = 2$$

$$f(x) = \frac{f(1-0) + f(1+0)}{2} = \frac{1+2}{2} = \frac{3}{2}$$

(ii) $x = 0$ is a discontinuous point in the end.

$$f(x) = \frac{f(0) + f(2)}{2} = \frac{0+2}{2} = 1.$$

Problems:

① Find the Fourier series for $f(x) = x^2$ in $-\pi \le x \le \pi$ and deduce that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

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(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

(iv) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$.

Soln:

$f(x) = x^2$ is an even function.

$b_n = 0$.

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{4}{n^2}(-1)^n.$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n \cos nx \rightarrow ①$$

(i) Put $x = \pi$ [$x = \pi$ is a continuous point. $\therefore f(x) = f(\pi) = \pi^2$] in ①,

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n \cos n\pi$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n (-1)^n$$

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^{2n}$$

$$\boxed{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\infty=\frac{\pi^2}{6}} \longrightarrow (2)$$

(ii) Put $x = 0$ [$x = 0$ is a continuous point.

$f(x) = f(0) = 0$] in (1),

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n \cos 0$$

$$-\frac{\pi^2}{3} = 4\left[\frac{-1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\cdots\right]$$

$$-\frac{\pi^2}{12} = -\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right]$$

$$\therefore \boxed{\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}\cdots=\frac{\pi^2}{12}} \longrightarrow (3)$$

(iii) Adding (2) and (3),

$$2\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty\right] = \frac{\pi^2}{6}+\frac{\pi^2}{12}$$

$$= \frac{2\pi^2+\pi^2}{12}$$

$$= \frac{3\pi^2}{12} = \frac{\pi^2}{4}$$

$$\therefore \boxed{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty=\frac{\pi^2}{8}}$$

(iv) Using Parseval's identity,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\frac{1}{\pi}\,2\int_{0}^{\pi}x^4\,dx = \frac{\left(\frac{4\pi^4}{9}\right)}{2}+\sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\right)^2$$

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$$\frac{2}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi} = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\frac{2\pi^4}{5} - \frac{2\pi^4}{9} = 16\sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{18\pi^4 - 10\pi^4}{45 \times 16} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\boxed{\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \infty = \frac{\pi^4}{90}}$$

(2) PT in $0 < x < l$,

Cla $x = \frac{l}{2} - \frac{4l}{\pi^2}\left[\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\left(\frac{3\pi x}{l}\right) + \cdots\right]$

& deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \cdots \infty = \frac{\pi^4}{96}$

(or)

Find half range cosine series for $f(x) = x$ in $(0, l)$ and deduce $\frac{1}{1^4} + \frac{1}{3^4} + \cdots \infty = \frac{\pi^4}{96}$.

Soln:

$$a_0 = l$$

$$a_n = \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

Using parseval's identity,

$$\frac{2}{l}\int_0^l [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

(3) Find half range sine series for $f(x) = x$ in $(0, l)$ & deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

Soln:

$$b_n = -\frac{2l}{n\pi}(-1)^n.$$

(4) Express $f(x) = (\pi - x)^2$ as a Fourier Series of Period 2π in the interval $0 < x < 2\pi$ and hence deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

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Soln:

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4}{n^2}, \quad b_n = 0.$$

$x = 0$ is a discontinuous point

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

(5) Find the Fourier Series of $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Ass

Soln:

$$a_0 = -\frac{\pi}{2}, \quad a_n = \frac{1}{\pi n^2}[(-1)^n - 1].$$

$$b_n = \frac{1}{n}[1 - 2(-1)^n] \qquad f(x) = \frac{f(-\pi) + f(\pi)}{2}$$

$$= \frac{-\pi + x}{2} = \frac{-\pi + 0}{2}$$

$x = 0 \Rightarrow f(0) = -\pi/2.$

Ass

(6) Find the Fourier Series of $f(x) = |x|$, $-\pi < x < \pi$ & hence deduce $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}$.

Soln:

$$a_0 = \pi$$

$$a_n = \frac{2}{\pi n^2}[(-1)^n - 1]$$

$$b_n = 0.$$

$x = 0$.