(2) Find the Fourier transform of

$$f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

. Hence deduce that

(i) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$ (ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$

Soln: The Fourier transform of $f(x)$ is,

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)\,(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-1}^{1} (1-|x|)\cos sx\, dx + \underbrace{\int_{-1}^{1} (1-|x|)\, i \sin sx\, dx}_{0} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 2 \int_0^1 (1-x)\cos sx\, dx$$

$$u = 1-x, \quad u' = -1, \quad u'' = 0$$

$$v = \cos sx, \quad v_1 = \frac{\sin sx}{s}, \quad v_2 = \frac{-\cos sx}{s^2}$$

$$F(s) = \frac{2}{\sqrt{2\pi}} \left\{ \cancel{(1-x)\frac{\sin sx}{s}} - \frac{\cos sx}{s^2} \right\}_0^1$$

$$F(s) = \frac{2}{\sqrt{2\pi}} \left\{ \frac{-\cos s}{s^2} + \frac{1}{s^2} \right\}$$

$$\boxed{F(s) = \frac{2}{\sqrt{2\pi}\, s^2} [1-\cos s]}$$

(i) Using inverse Fourier transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$1-|x| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \frac{2}{\sqrt{2\pi}\, s^2} (1-\cos s) \right\} (\cos sx - i \sin sx)\, ds$$

$$= \frac{\cancel{2}}{\cancel{2}\pi} \left[\int_{-\infty}^{\infty} \frac{(1-\cos s)}{s^2} \cos sx\, ds - \underbrace{\int_{-\infty}^{\infty} \frac{(1-\cos s)}{s^2}\, i \sin sx\, ds}_{0} \right]$$

$$1-|x| = \frac{2}{\pi} \int_0^{\infty} \frac{1-\cos s}{s^2} \cos sx\, ds$$

Put $x = 0$, $s = 2t$

$\Rightarrow ds = 2\,dt$

$$1 = \frac{2}{\pi} \int_0^{\infty} \frac{1-\cos 2t}{(2t)^2}\, 2\, dt$$

$$1 = \frac{\cancel{2}}{\pi} \int_0^{\infty} \frac{1-\cos 2t}{\cancel{4}t^2}\, \cancel{2}\, dt$$

$$\boxed{\frac{1-\cos 2t}{2} = \sin^2 t}$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{2\sin^2 t}{t^2}\, dt$$

$$1 = \frac{2}{\pi} \int_0^{\infty} \frac{\sin^2 t}{t^2}\, dt$$

$$\therefore \boxed{\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}}$$

(ii) Using Parseval's identity,

$$\int_{-\infty}^{\infty} [F(s)]^2 ds = \int_{-\infty}^{\infty} [f(x)]^2 dx$$

$$\int_{-\infty}^{\infty} \left[\frac{2}{\sqrt{2\pi}\, s^2}(1-\cos s)\right]^2 ds = \int_{-1}^{1} (1-|x|)^2 dx$$

$$\frac{\cancel{4}}{\cancel{2}\pi} \int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds = 2\int_{0}^{1} (1-x)^2 dx$$

$$\frac{2}{\pi} \cdot 2 \int_{0}^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds = 2\int_{0}^{1} (1+x^2-2x)\, dx$$

$$\frac{4}{\pi} \int_{0}^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds = 2\left[x+\frac{x^3}{3} - \cancel{2}\frac{x^2}{\cancel{2}}\right]_0^1$$

$$= 2\left[1+\frac{1}{3}-1\right] = \frac{2}{3}$$

$$\therefore \int_{0}^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds = \frac{\cancel{2}}{3} \times \frac{\pi}{\cancel{4}_2} = \frac{\pi}{6}$$

Put $s = 2t \Rightarrow ds = 2\, dt$

$$\int_{0}^{\infty} \left(\frac{1-\cos 2t}{4t^2}\right)^2 2\, dt = \frac{\pi}{6}$$

$$\int_{0}^{\infty} \left(\frac{2\sin^2 t}{4t^2}\right)^2 2\, dt = \frac{\pi}{6}$$

$$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^4 \times \frac{\cancel{4}}{\cancel{16}} \times \cancel{2}\, dt = \frac{\pi}{6} \Rightarrow \boxed{\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \pi/3}$$