(3) Find the Fourier transform of

$$f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a > 0. \end{cases}$$

Hence deduce that

(i) $\displaystyle\int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}$

(ii) $\displaystyle\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$

Solution:

The Fourier transform of $f(x)$ is,

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)(\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} (a^2 - x^2)\cos sx\, dx + \cancelto{0}{\int_{-a}^{a} (a^2 - x^2)\, i\sin sx\, dx} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 2 \int_{0}^{a} (a^2 - x^2)\cos sx\, dx$$

$$F(s) = \frac{2}{\sqrt{2\pi}}\left\{(a^2-x^2)\frac{\sin sx}{s} - 2x\frac{\cos sx}{s^2} + 2\frac{\sin sx}{s^3}\right\}_0^a$$

$u = a^2 - x^2$, $v = \cos sx$

$u' = -2x$, $v_1 = \frac{\sin sx}{s}$

$u'' = -2$, $v_2 = \frac{-\cos sx}{s^2}$

$u''' = 0$, $v_3 = \frac{-\sin sx}{s^3}$

$$= \frac{2}{\sqrt{2\pi}}\left\{-\frac{2a\cos sa}{s^2} + \frac{2\sin sa}{s^3}\right\}$$

$$= \frac{4}{\sqrt{2\pi}}\left\{-\frac{a\cos sa}{s^2} + \frac{\sin sa}{s^3}\right\}$$

$$F(s) = \frac{4}{\sqrt{2\pi}\,.s^3}\left[\sin sa - as\cos sa\right]$$

(i) Using inverse Fourier transform,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}\, s^3}\left[\sin sa - as\cos sa\right]\left[\cos sx - i\sin sx\right] ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}\, s^3}(\sin sa - as\cos sa)\cos sx\, ds$$

$$- \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}\, s^3}(\sin sa - as\cos sa)\, i\sin sx\, ds \to 0$$

$$f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin sa - as\cos sa}{s^3}\cos sx\, ds$$

$$a^2 - x^2 = \frac{2}{\pi} \cdot 2 \int_0^\infty \frac{\sin sa - as \cos sa}{s^3} \cos sx \, ds$$

Put $s = t$, $a = 1$, $x = 0$

$$1 - 0 = \frac{4}{\pi} \int_0^\infty \frac{\sin t - t \cos t}{t^3} \, dt$$

$$\therefore \boxed{\int_0^\infty \frac{\sin t - t \cos t}{t^3} \, dt = \frac{\pi}{4}}$$

(ii) Using Parseval's identity,

$$\int_{-\infty}^{\infty} [F(s)]^2 ds = \int_{-\infty}^{\infty} [f(x)]^2 dx$$

$$\int_{-\infty}^{\infty} \left[\frac{4}{\sqrt{2\pi}\, s^3} [\sin sa - as \cos sa] \right]^2 ds = \int_{-a}^{a} (a^2 - x^2)^2 dx$$

$$\frac{16}{\cancel{2}\pi} \times \cancel{2} \int_0^\infty \left(\frac{\sin sa - as \cos sa}{s^3} \right)^2 ds = 2 \int_0^a (a^2 - x^2)^2 dx$$

$$\frac{16}{\pi} \int_0^\infty \left(\frac{\sin sa - as \cos sa}{s^3} \right)^2 ds = 2 \int_0^a (a^4 + x^4 - 2a^2 x^2) dx$$

$$= 2 \left[a^4 x + \frac{x^5}{5} - 2a^2 \frac{x^3}{3} \right]_0^a$$

$$= 2 \left[a^5 + \frac{a^5}{5} - \frac{2a^5}{3} \right]$$

$$= 2 \left(\frac{15a^5 + 3a^5 - 10a^5}{15} \right)$$

$$\frac{\cancel{16}}{\pi}\int_0^\infty\left(\frac{\sin sa - as\cos sa}{s^3}\right)^2 ds = \frac{\cancel{16}\,a^5}{15}$$

Put $a=1, s=t \Rightarrow$

$$\boxed{\int_0^\infty\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}}$$