FOURIER SINE TRANSFORM & FOURIER COSINE TRANSFORM

1. ★ Fourier cosine transform of $f(x)$ is

$$F_c[f(x)] = F_c(s) = \frac{2}{\sqrt{2\pi}} \int_0^\infty f(x) \cos sx \, dx$$

2. ★ Inverse fourier cosine transform of $f(x)$ is

$$f(x) = \frac{2}{\sqrt{2\pi}} \int_0^\infty F_c(s) \cos sx \, ds$$

Fourier cosine transform and Inverse Fourier cosine transform are jointly called as Fourier cosine transform pair.

3. ★ Fourier sine transform of $f(x)$ is,

$$F_S[f(x)] = F_S(s) = \frac{2}{\sqrt{2\pi}} \int_0^\infty f(x) \sin sx \, dx$$

4. ★ Inverse Fourier sine transform of $f(x)$ is

$$f(x) = \frac{2}{\sqrt{2\pi}} \int_0^\infty F_S(s) \sin sx \, ds$$

Fourier sine transform and Inverse Fourier sine transform are jointly called as Fourier sine transform pair.

4

Properties :

* FST and FCT are linear

$$F_c[a\,f(x) + b\,g(x)] = a\,F_c[f(x)] + b\,F_c[g(x)]$$
$$= a\,F_c(s) + b\,F_c(s).$$

* $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(a+s) + F_s(a-s)]$
* $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(a+s) + F_c(a-s)]$
* $F_s[f(x)\sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$
* $F_s[f(x)\cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$.
* $F_c[f(ax)] = \frac{1}{a}F_c(s/a)$
* $F_s[f(ax)] = \frac{1}{a}F_s(s/a)$
* $F_s[x\,f(x)] = -\frac{d}{ds}F_c(s)$
* $F_c[x\,f(x)] = \frac{d}{ds}F_s(s)$
* $F_c[f'(x)] = s\,F_s(s) - f(0).\sqrt{\frac{2}{\pi}}$
* $F_s[f'(x)] = -s\,F_c(s)$

5 * Parseval's identity for FCT is

$$\int_0^\infty [f(x)]^2\,dx = \int_0^\infty [F_c(s)]^2\,ds$$

6 * Parseval's identity for FST is.

$$\int_0^\infty [f(x)]^2\,dx = \int_0^\infty [F_s(s)]^2\,ds.$$

7.* $\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_c(s)\, G_c(s)\, ds$

8.* $\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_s(s)\, G_s(s)\, ds$

Problems:

(1) Find FCT of $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$

Soln:

FCT of $f(x)$ is

$$F_c(s) = F_c[f(x)] = \frac{2}{\sqrt{2\pi}} \int_0^\infty f(x) \cos sx\, dx \quad (1)$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a \cos x \cos sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a \cos sx \cos x\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a \frac{1}{2} [\cos(s+1)x + \cos(s-1)x]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s-1} \right]$$

(2) Find FST of $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$

Soln:

FST of $f(x)$ is,

$$F_s(s) = F_s(f(x)) = \frac{2}{\sqrt{2\pi}} \int_0^\infty f(x) \sin sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a \sin sx \sin x \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a -\frac{1}{2} \left[\cos(s+1)x - \cos(s-1)x\right] dx$$

$$= \frac{-1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)a}{s+1} - \frac{\sin(s-1)a}{s-1}\right]$$

(3) Find FST of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

UQ

soln:

$$F_s(s) = \frac{2}{\sqrt{2\pi}} \int_0^\infty f(x) \sin sx \, dx$$

(2)

$$= \frac{2}{\sqrt{2\pi}} \left[\int_0^1 x \sin sx \, dx + \int_1^2 (2-x) \sin sx \, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{s^2} \left[2 \sin s - \sin 2s\right]$$

$(\because \sin 2\theta = 2 \sin\theta \cos\theta)$

not necessary

$$= \sqrt{\frac{2}{\pi}} \frac{1}{s^2} \left[2 \sin s - 2 \sin s \cos s\right]$$

$$= \frac{2\sqrt{2}}{\sqrt{\pi}} \cdot \frac{1}{s^2} \sin s (1 - \cos s)$$

(4) Find FST of $f(x) = \begin{cases} 0, & 0 < x < a \\ x, & a < x < b \\ 0, & x > b \end{cases}$

soln:

$$F_s(s) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{s^2} \left[-bs \cos bs + \sin bs + as \cos as - \sin as\right]$$