Solving Differential Equation using Laplace Transform

Shifting property of Unilateral Laplace Transform.

★ $L\left[\frac{d}{dt} x(t)\right] = s\,X(s) - x(0^-)$

★ $L\left[\frac{d^2}{dt^2} x(t)\right] = s^2 X(s) - s\,x(0^-) - x'(0^-)$

★ $L\left[\frac{d^3}{dt^3} x(t)\right] = s^3 X(s) - s^2 x(0^-) - s\,x'(0^-) - x''(0^-)$

1) Solve using Differential Equation $\frac{d}{dt} y(t) + 5y(t) = x(t)$ with initial condition $y(0^-) = -2$ and i/p $x(t) = 3e^{-2t} u(t)$

$$\frac{d}{dt} y(t) + 5\,y(t) = x(t)$$

$$s\,Y(s) - y(0^-) + 5Y(s) = X(s)$$

$$s\,Y(s) + 2 + 5\,Y(s) = \frac{3}{s+2}$$

$$Y(s)\,[s+5] + 2 = \frac{3}{s+2}$$

$$Y(s)\,[s+5] = \frac{3}{s+2} - 2$$

$$Y(s) = \frac{3}{(s+2)[s+5]} - \frac{2}{(s+5)}$$

$$\therefore \frac{3}{(S+2)(S+5)} = \frac{A}{S+2} + \frac{B}{S+5}$$

$$3 = A(S+5) + B(S+2)$$

put $S=-5$

$3 = B(-3)$

$B = -1$

put $S=-2$

$3 = A(3)$

$A = 1$

$$Y(S) = \left[\frac{1}{S+2} - \frac{1}{S+5}\right] - \frac{2}{S+5}$$

$$Y(S) = \frac{1}{S+2} - \frac{3}{S+5}$$

$$\therefore y(t) = e^{-2t}u(t) - 3e^{-5t}u(t).$$

System Transfer Function :-

$$y(t) = x(t) * h(t)$$

$$Y(S) = X(S) * H(S)$$

$$H(S) = \frac{Y(S)}{X(S)} \rightarrow \text{system Transfer Function}$$

Freq Response :-

By substituting $S = j\omega$ in $H(S)$ we can get the freq response $H(j\omega)$.

① The input output relation of a system at initial rest is given by $\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 3y(t) = \frac{d}{dt}x(t) + 2x(t)$. Find system transfer function, freq response and Impulse response?

$$s^2 Y(s) + 4sY(s) + 3Y(s) = sX(s) + 2X(s)$$

$$Y(s)\left[s^2+4s+3\right] = X(s)\left[s+2\right]$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^2+4s+3}$$

Freq Response :-

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega+2}{(j\omega)^2+4j\omega+3}$$

Impulse Response :-

$$H(s) = \frac{s+2}{s^2+4s+3}$$

$$\frac{s+2}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1}$$

$$s+2 = A(s+1) + B(s+3)$$

sub $s=-1$ | sub $s=-3$

$1 = 2B$ | $-1 = -2A$

$\boxed{B = 1/2}$ | $\boxed{A = 1/2}$

$$H(s) = \frac{1}{2(s+3)} + \frac{1}{2(s+1)}$$

$$h(t) = \frac{1}{2} L^{-1}\left(\frac{1}{s+3}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s+1}\right)$$

$$h(t) = \frac{1}{2} e^{-3t} u(t) + \frac{1}{2} e^{-t} u(t)$$

(2) The Differential Equation of a system is given as

$$\frac{d^2}{dt^2} y(t) + 3\frac{d}{dt} y(t) + 2y(t) = x(t)$$ with Initial conditions $y(0^+) = 3$, $y'(0^+) = -5$. Determine the o/p for the i/p $x(t) = 2u(t)$

$$s^2 Y(s) - s\,y(0^-) - y'(0^-) + 3[sY(s) - y(0^-)] + 2Y(s) = X(s)$$

$$s^2 Y(s) - 3s + 5 + 3[sY(s) - 3] + 2Y(s) = X(s)$$

$$s^2 Y(s) - 3s + 5 + 3sY(s) - 9 + 2Y(s) = X(s)$$

$$Y(s)[s^2 + 3s + 2] - 3s - 9 + 5 = X(s)$$

$$Y(s)[s^2 + 3s + 2] = 2/s + [3s + 4]$$

$$Y(s) = \frac{2 + 3s^2 + 4s}{s(s+1)(s+2)}$$

$$2 + 3s^2 + 4s = A(s+1)(s+2) + B(s)(s+2) + C(s)(s+1)$$

$s = -1$ $\boxed{B = -1}$

$s = -2$ $\boxed{C = 3}$

$s = 0$ $\boxed{A = 1}$

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} + \frac{3}{s+2}$$

$$= L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + 3L^{-1}\left(\frac{1}{s+2}\right)$$

$$\therefore y(t) = u(t) - e^{-t}u(t) + 3e^{-2t}u(t)$$

3) $\frac{d^2}{dt^2} y(t) - \frac{d}{dt} y(t) - 2y(t) = x(t)$

$s^2 Y(s) - sY(s) - 2Y(s) = X(s)$

$(s^2 - s - 2) Y(s) = X(s)$

$\therefore \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2} = \frac{1}{(s-2)(s+1)}$

$\frac{A}{s-2} + \frac{B}{s+1} = \frac{1}{(s-2)(s+1)}$

$1 = A(s+1) + B(s-2)$

$s = -1$ put $s = 2$

$B(-3) = 1$ $3A = 1$

$\boxed{B = -1/3}$ $\boxed{A = 1/3}$

$H(s) = \frac{1}{3(s-2)} - \frac{1}{3(s+1)}$

$= \frac{1}{3} L^{-1}\left(\frac{1}{s-2}\right) - \frac{1}{3} L^{-1}\left(\frac{1}{s+1}\right)$

$h(t) = \frac{1}{3} e^{2t} u(t) - \frac{1}{3} e^{-t} u(t)$

4) $H(s) = \frac{s}{s^2+5s+6}$ and $x(t) = e^{-t} u(t)$. Determine the o/p assuming zero Initial conditions.

$Y(s) = \frac{s}{s^2+5s+6}$ $X(s) = \frac{1}{s+1}$

$Y(s) = \frac{s}{(s+3)(s+2)(s+1)}$

$S = A(S+2)(S+1) + B(S+3)(S+1) + C(S+3)(S+2)$

Put $S=-2$

$-2 = B(1)(-1)$

$\boxed{B=2}$

Put $S=-3$

$-3 = 2A$

$\boxed{A = -3/2}$

Put $S=-1$

$-1 = C(2)(1)$

$\boxed{C = -1/2}$

$$Y(S) = \frac{-3}{2(S+3)} + \frac{2}{S+2} - \frac{1}{2(S+1)}$$

$$= -\frac{3}{2} L^{-1}\left(\frac{1}{S+3}\right) + 2 L^{-1}\left(\frac{1}{S+2}\right) - \frac{1}{2} L^{-1}\left(\frac{1}{S+1}\right)$$

$$y(t) = -\frac{3}{2} e^{-3t} u(t) + 2 e^{-2t} u(t) - \frac{1}{2} e^{-t} u(t)$$

5) Find the Impulse response of the system :-

$$RC \frac{d}{dt} y(t) + y(t) = x(t)$$

$$RC\, sY(S) + Y(S) = X(S)$$

$$Y(S)(RCs+1) = X(S)$$

$$H(S) = \frac{Y(S)}{X(S)} = \frac{1}{(RCs+1)}$$

$$y(t) = L^{-1}\left[\frac{1}{RCs+1}\right]$$

$$= \frac{1}{RC}\left[L^{-1}\left(\frac{1}{S+1/RC}\right)\right]$$

$$\therefore y(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

(6) Find the o/p of the system :- $h(t) = u(t)$, $x(t) = e^{-2t} \cdot u(t)$

$$H(S) = 1/s \qquad X(S) = \frac{1}{S+2}$$

$$H(S) = \frac{Y(S)}{X(S)}$$

$$Y(S) = H(S) \cdot X(S)$$

$$= 1/s \cdot \frac{1}{S+2}$$

$$= \frac{A}{S} + \frac{B}{s+2} \Rightarrow A(s+2) + B(S)$$

$S = 0$

$1 = A(2)$

$$\boxed{A = 1/2}$$

$S = -2$

$1 = A(0) + B(-2)$

$$\boxed{B = -1/2}$$

$$Y(S) = \frac{1}{2(S)} - \frac{1}{2(S+2)}$$

$$= 1/2 \, L^{-1}(1/s) - 1/2 \, L^{-1}\left(\frac{1}{S+2}\right)$$

$$y(t) = 1/2 \, u(t) - 1/2 \, e^{-2t} \, u(t)$$