UNIT – III

PARTIAL DIFFERENTIAL EQUATIONS

Homogeneous linear Partial Differential Equations:

A homogeneous linear partial differential equation is of the form,

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = f(x,y) \rightarrow (1)$$

where $a_0, a_1, a_2, \cdots a_n$ are constants.

Here $\frac{\partial}{\partial x} = D$, $\frac{\partial}{\partial y} = D'$.

$\therefore$ (1) becomes,

$$a_0 D^n z + a_1 D^{n-1} D' z + a_2 D^{n-2} D'^2 z + \cdots + a_n D'^n z = f(x,y) \rightarrow (2)$$

Solution of Homogeneous Linear Pde:

The Complete solution of (2) is,

z = Complementary function + Particular integral

i.e., $z = C.F + P.I$

To find C.F ;

The C.F is the solution of equation,

$$[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n] z = 0$$

In the above equation, put $D \rightarrow m$ & $D' \rightarrow 1$

then we get,

$[a_0 m + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n] = 0$ which is called auxilliary equations.

For $n = 2$, $a_0 m^2 + a_1 m + a_2 = 0$

m_1, m_2 are two roots of the above eqns.

Case (i): $m_1 = m_2 = m$

$C.F = f_1(y + mx) + x f_2(y + mx)$

Case (ii): m_1 & m_2 are different

$C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$

~~Case (ii)~~:

Problems:

(1) Solve: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0$

Soln:

A.E is $(D^2 + DD' - 2D'^2) z = 0$

~~$m^2 + m - 2 = 0$~~

Put $D = m$ & $D' = 1$

$m^2 + m - 2 = 0$

$m_1 = 1,\ m_2 = -2$

C.F: Two roots are different.

$C.F = f_1(y + x) + f_2(y - 2x)$

P.I: RHS $= 0$.

P.I $= 0$.

Solution: $z = C.F + P.I$

$z = f_1(y + x) + f_2(y - 2x)$

(2) Solve $(D^2 - 5DD' + 6D'^2)z = 0$

Soln: The A.E is,

$m^2 - 5m + 6 = 0 \qquad (D \to m,\ D' \to 1)$

$m_1 = 3,\ m_2 = 2$

The roots are real & different.

$C.F = f_1(y+3x) + f_2(y+2x)$.

$RHS = 0 \Rightarrow P.I = 0$.

Solution: $z = C.F + P.I$

$z = f_1(y+3x) + f_2(y+2x)$

(3) Solve : $(D^2 - 2DD' + D'^2)z = 0$

Soln: The A.E is,

$m^2 - 2m + 1 = 0 \qquad (D \to m,\ D' \to 1)$

$(m-1)^2 = 0$

$m = 1, 1$

The roots are real & equal.

$C.F = f_1(y+x) + x f_2(y+x)$

$P.I = 0$

$\therefore z = C.F + P.I$

$= f_1(y+x) + x f_2(y+x)$

(4) Solve $(D^2 + D'^2)z = 0$.

To find P.I :

Type I : RHS $= f(x,y) = e^{ax+by}$

$$P.I = \frac{1}{\phi(D,D')} e^{ax+by}$$

$D = a \;,\; D' = b$

$$P.I = \frac{1}{\phi(a,b)} e^{ax+by} \quad \text{provided } \phi(a,b) \neq 0$$

If $\phi(a,b) = 0$, then differentiate the denominator w.r.t D and put 'x' in numerator.

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(1) Solve : $(D^2 - DD' - 20D'^2)\, z = e^{5x+y}$

Soln :

A.E is,

$m^2 - m - 20 = 0 \quad (D \to m \,,\, D' \to 1)$

$m_1 = 5 \;,\; m_2 = -4$

Two roots are real & different.

-20

$-5+4$

$C.F = f_1(y+5x) + f_2(y-4x)$.

$$P.I = \frac{e^{5x+y}}{D^2 - DD' - 20D'^2}$$

$D \to a \to 5$

$D' \to b \to 1$

$$= \frac{e^{5x+y}}{25-5-20}$$

$$= \frac{e^{5x+y}}{0} = \frac{x e^{5x+y}}{2D - D'}$$

$$= \frac{x e^{5x+y}}{10-1} = \frac{x e^{5x+y}}{9}$$

$z = C.F + P.I$

$z = f_1(y+5x) + f_2(y-4x) + \frac{xe^{5x+y}}{9}$

2) Solve: $(2D^2 - 2DD' + D'^2)z = 2e^{3y} + e^{x+y}$

Soln:

The A.E is,

$2m^2 - 2m + 1 = 0$

$m_1 = 0.5 + 0.5i, \quad m_2 = 0.5 - 0.5i$

$\therefore C.F = f_1[y + (0.5+0.5i)x] + f_2[y + (0.5-0.5i)x]$

$P.I_1 = \frac{2e^{3y}}{2D^2 - 2DD' + D'^2}$ $\quad D \to a \to 0$, $D' \to b \to 3$

$\boxed{P.I_1 = \frac{2e^{3y}}{9}}$

$P.I_2 = \frac{e^{x+y}}{2D^2 - 2DD' + D'^2}$ $\quad D \to a \to 1$, $D' \to b \to 1$

$= \frac{e^{x+y}}{2-2+1} = e^{x+y}$

$\boxed{P.I_2 = e^{x+y}}$

$P.I = P.I_1 + P.I_2$

$z = C.F + P.I$

$z = f_1[y + (0.5+0.5i)] + f_2(y + 0.5 - 0.5i) + \frac{2e^{3y}}{9} + e^{x+y}$

(3) Solve : $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$

Soln:

The A.E is,

$m^2 - 4m + 4 = 0$

$m = 2, 2$

$C.F = f_1(y+2x) + x f_2(y+2x)$

$P.I = \frac{x^2 e^{2x+y}}{2}$

$z = C.F + P.I$

$z = f_1(y+2x) + x f_2(y+2x) + \frac{x^2 e^{2x+y}}{2}$.

Type II : RHS $= f(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

$P.I = \frac{\sin(ax+by)}{\phi(D,D')}$ (or) $\frac{\cos(ax+by)}{\phi(D,D')}$

Replace $D^2 \to -a^2$, $D'^2 \to -b^2$ and $DD' \to -ab$.

(1) Solve : $(D^2 + DD' - 6D'^2)z = \cos(2x+3y)$

Soln:

A.E is,

$m^2 + m - 6 = 0$

$m_1 = 2$, $m_2 = -3$

$C.F = f_1(y+2x) + f_2(y-3x)$.

$$P.I = \frac{\cos(2x+3y)}{D^2 + DD' - 6D'^2}$$

Replace $D^2 \to -a^2$, $DD' \to -ab$, $D'^2 \to -b^2$

$a = 2$, $b = 3$.

$$P.I = \frac{\cos(2x+3y)}{-4-6+54}$$

$$= \frac{\cos(2x+3y)}{44}$$

$\therefore$ The solution is,

$$y = C.F + P.I$$

$$y = f_1(y+2x) + f_2(y-3x) + \frac{\cos(2x+3y)}{44}.$$

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(9) Solve : $(2D^2 - 5DD' + 2D'^2)z = \sin(2x+y) + e^{2x+y}$

Soln:

Put $D = m$, $D' = 1$

The A.E is,

$$2m^2 - 5m + 2 = 0$$

$$2m^2 - m - 4m + 2 = 0$$

$$m(2m-1) - 2(2m-1) = 0$$

$$2m - 1 = 0 \quad , \quad m - 2 = 0$$

$$m = 1/2 \quad , \quad m = 2$$

$$C.F = f_1\left(y + \frac{1}{2}x\right) + f_2(y + 2x)$$

$$P.I_1 = \frac{1}{2D^2 - 5DD' + 2D'^2} \sin(2x+y)$$

$$= \frac{1}{-8+10-2} \sin(2x+y)$$

[Here $a = 2$, $b = 1$
$D^2 = -a^2 = -4$
$DD' = -2$, $D'^2 = -1$]

$$= \frac{1}{0} \sin(2x+y)$$

$$= \frac{x}{4D - 5D'} \sin(2x+y)$$

$$= \frac{xD}{4D^2 - 5DD'} \sin(2x+y)$$

$$= \frac{xD\sin(2x+y)}{4(-4) - 5(-2)}$$

$$= \frac{x\cos(2x+y) \cdot 2}{-16+10}$$

$$= -\frac{2x\cos(2x+y)}{6}$$

$$P.I_1 = \frac{-x}{3}\cos(2x+y).$$

$$P.I_2 = \frac{1}{2D^2 - 5DD' + 2D'^2} e^{2x+y}$$

$$P.I_2 = \frac{1}{8-10+2} e^{2x+y} \qquad [\text{Here } D \to 2,\ D' \to 1]$$

$$= \frac{1}{0} e^{2x+y}$$

$$= \frac{x}{4D-5D'} e^{2x+y} = \frac{x}{8-5} e^{2x+y}$$

$$P.I_2 = \frac{x}{3} e^{2x+y}$$

$\therefore$ The general solution is,

$$z = C.F + P.I_1 + P.I_2.$$

$$= f_1\left(y+\tfrac{1}{2}x\right) + f_2\left(y+2x\right) + \frac{x}{3} e^{2x+y} - \frac{x}{3}\cos(2x+y)$$

(10) Solve : $(D^2 + DD' - 6D'^2)z = \cos(2x+y) + e^{x-y}$.

Soln:

Put $m = D$, $D' = 1$.

The A.E is $m^2 + m - 6 = 0$

$m = -3$, $m = 2$.

$$C.F = f_1(y-3x) + f_2(y+2x)$$

$$P.I_1 = \frac{1}{D^2 + DD' - 6D'^2}\cos(2x+y).$$

$$= \frac{1}{-4-2+6}\cos(2x+y) \qquad [\text{Here } a=2,\ b=1,\ D^2 = -4,\ DD' = -2,\ D'^2 = -1]$$

$$= \frac{1}{0}\cos(2x+y)$$

$$= \frac{x}{2D+D'}\cos(2x+y)$$

$$= \frac{xD}{2D^2 + DD'}\cos(2x+y)$$

$$= \frac{x\,[-\sin(2x+y)\cdot 2]}{-8-2}$$

$$= +\frac{2x}{10}\sin(2x+y) = \frac{x}{5}\sin(2x+y)$$