Half Range Expansions

*) The half range cosine series in the interval $(0, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

*) The half range sine series in the interval $(0, l)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Problem:

Find the sine series of $f(x) = x$ in $(0, l)$.

Solution:

Step 1: The Fourier series of $f(x)$ in $(0, l)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Step 2: To find b_n

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\int_0^l x\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[x\cdot\left(\frac{-l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right) - 1\cdot\left(\frac{-l^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-l^2}{n\pi}\cos(n\pi) + \frac{l^2}{n^2\pi^2}\sin(n\pi) - (0+0)\right]$$

$$= \frac{2}{l}\left[\frac{-l^2}{n\pi}(-1)^n\right]$$

$$\boxed{b_n = \frac{+2l}{n\pi}(-1)^{n+1}}$$

Step 3 : The required Fourier Sine Series

Sub. b_n in Step 1.

$$f(x) = \sum_{n=1}^{\infty} \frac{2l}{n\pi} (-1)^{n+1} \sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{l}\right)$$

2) Find the sine series for 1) $f(x) = x^2$ in $(0, l)$

2) $f(x) = l - x$ in $(0, l)$

Solution:

Step 1: The Fourier Sine Series of $f(x)$ in $(0,l)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Step 2: To find b_n

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \int_0^l x^2 \sin\left(\frac{n\pi x}{l}\right) dx$$

$u = x^2$ | $dv = \sin\left(\frac{n\pi x}{l}\right) dx$

$u' = 2x$ | $v = \frac{-l}{n\pi} \cos\left(\frac{n\pi x}{l}\right)$

$u'' = 2$ | $v_1 = \frac{-l^2}{n^2\pi^2} \sin\left(\frac{n\pi x}{l}\right)$

$v_2 = \frac{+l^3}{n^3\pi^3} \cos\left(\frac{n\pi x}{l}\right)$

$$b_n = \frac{2}{l}\left[x^2\left(\frac{-l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right) - (2x)\left(\frac{-l^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{l}\right) + 2\left(\frac{l^3}{n^3\pi^3}\right)\cos\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left(\frac{-l^3}{n\pi}(-1)^n + \frac{(2l^3)}{n^2\pi^2}(0) + \frac{2l^3}{n^3\pi^3}(-1)^n\right) - \left(\frac{2l^3}{n^3\pi^3}\right)\right]$$

$$\frac{2}{l}\left[\frac{l^3}{n\pi}(-1)^{n+1} + \frac{2l^3}{n^3\pi^3}\left[(-1)^n - 1\right]\right]$$

$$= \frac{2}{l}\left[l^3(-1)^n\left[\frac{-1}{n\pi}+\frac{2}{n^3\pi^3}\right] - \frac{2l^3}{n^3\pi^3}\right] \Bigg/ \; b_n = \frac{2}{l}\left[\frac{l^3}{n\pi}(-1)^{n+1} + \frac{2l^3}{n^3\pi^3}\left((-1)^n-1\right)\right]$$

$$= \frac{2}{l}\left[l^3(-1)^n\left[\frac{-n^2\pi^2+2}{n^3\pi^3}\right] - \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{2}{l(n^3\pi^3)}\left[(-1)^n(l^3) - 2l^3\right] = \frac{2l^3}{l n^3\pi^3}\left((-1)^n - 2\right)$$

$$= \frac{2l^2}{n^3\pi^3}\left[(-1)^n - 2\right] \qquad \boxed{b_n = \frac{2l^2}{n^3\pi^3}\left[(-1)^n-2\right]}$$

$$b_n = \begin{cases} \dfrac{-6l^2}{n^3\pi^3}, & n \text{ is odd} \\ \dfrac{-2l^2}{n^3\pi^3}, & n \text{ is even} \end{cases}$$

Step 3: $f(x) = \frac{2}{l}\sum_{n=1}^{\infty}\left[\frac{l^3}{n\pi}(-1)^{n+1} + \frac{2l^3}{n^3\pi^3}\left((-1)^n-1\right)\right]\sin\left(\frac{n\pi x}{l}\right)$

$$f(x) = \sum_{n=1,3,5}^{\infty}\left(\frac{-6l^2}{n^3\pi^3}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \sum_{n=2,4,6,}^{\infty}\left(\frac{-2l^2}{n^3\pi^3}\right)\sin\left(\frac{n\pi x}{l}\right)$$

3) Find the Sine Series for $f(x) = l - x$ in $(0, l)$

Solution:

Step 1: The Fourier Sine Series of $f(x)$ in $(0,l)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Step 2: To find b_n

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\int_0^l (l-x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$u = l - x$ | $dv = \sin\left(\frac{n\pi x}{l}\right)dx$

$u' = -1$ | $v = \frac{-l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)$

$u'' = 0$ | $v_1 = \frac{-l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)$

$$b_n = \frac{2}{l}\left[(l-x)\left(\frac{-l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right) + \left(\frac{-l^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[(0+0) + \frac{l^2}{n\pi}(1)\right]$$

$$= \frac{2l^2}{l n\pi} = \frac{2l}{n\pi} \qquad \boxed{b_n = \frac{2l}{n\pi}}$$

Step 3: The required Sine Series

Substituting b_n in Step 1

$$f(x) = \sum_{n=1}^{\infty} \frac{2l}{n\pi} \sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) //$$

In the Interval $(0,\pi)$

1) Find the half range cosine series of $f(x) = x(\pi - x)$ in $(0,\pi)$

Solution:

Step 1:

The half range cosine series of the function $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx).$$

Step 2: To find a_0

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (x\pi - x^2)\,dx$$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi} \qquad \boxed{a_0 = \frac{\pi^2}{3}}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi^3}{2} - \frac{\pi^3}{3}\right) - (0-0)\right] = \frac{2}{\pi}\left[\frac{\pi^3}{6}\right] \quad \cancel{a_0 = \frac{\pi^2}{6}}$$

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Step 3: To find a_n

$u = x\pi - x^2$ $\quad dv = \cos nx\, dx$

$u' = \pi - 2x$ $\quad v = \frac{1}{n}\sin nx$

$u'' = -2$ $\quad v_1 = \frac{-1}{n^2}\cos nx$

$u''' = 0$ $\quad v_2 = \frac{-1}{n^3}\sin nx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(x\pi - x^2)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[(x\pi - x^2)\left(\frac{1}{n}\sin nx\right) - (\pi - 2x)\left(\frac{-1}{n^2}\cos nx\right) + (-2)\left(\frac{-1}{n^3}\sin nx\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 - \left(\frac{\pi}{n^2}\right)(-1)^n + 0\right) - \left(\frac{\pi}{n^2}\right)(1)\right] = \frac{2}{\pi}\left[\frac{-\pi}{n^2}\left((-1)^n + 1\right)\right]$$

$$a_n = \frac{-2}{n^2}\left[(-1)^n + 1\right]$$

$$a_n = \begin{cases} 0 & , n \text{ is odd} \\ \frac{-4}{n^2} & , n \text{ is even} \end{cases}$$

Step 4: The required Fourier Cosine Series

$$f(x) = \frac{\pi^2/3}{2} + \sum_{n=2,4,6}^{\infty}\left(\frac{-4}{n^2}\right)\cos(nx)$$

$$f(x) = \frac{\pi^2}{6} - 4\sum_{n=2,4,6}^{\infty}\frac{\cos(nx)}{n^2}$$

HW

$f(x) = x$

$f(x) = x^2$

$(0, \pi)$

Cosine Series

2.) Find the half range Cosine Series for the function

$$f(x) = \begin{cases} x & , 0 \le x < \pi/2 \\ \pi - x & , \pi/2 \le x < \pi \end{cases}$$

Solution:

Step 1: The Cosine Series of $f(x)$ in $(0, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Step 2: To find a_0

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\, dx + \int_{\pi/2}^{\pi} \pi - x\, dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi^2/4}{2} - 0\right) + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2/4}{2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi^2}{8} + \pi^2 - \frac{2\pi^2}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi^2 + 8\pi^2 - 8\pi^2}{8}\right] \qquad \boxed{a_0 = \frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left[\frac{2\pi^2}{8}\right] = \frac{\pi}{2}$$

<u>Step 3</u>: To find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left\{\left[x\left(\frac{1}{n}\sin nx\right) - \left(\frac{-1}{n^2}\cos nx\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(\frac{1}{n}\sin nx\right) + \left(\frac{-1}{n^2}\cos nx\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\left[\frac{\pi}{2n}\cancel{\sin\left(\frac{n\pi}{2}\right)} + \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right)\right] - \left(0 + \frac{1}{n^2}\right) + \left[\left(0 - \frac{1}{n^2}(-1)^n\right) - \left(\cancel{\frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right)} + \frac{1}{n^2}\cos n\left(\frac{\pi}{2}\right)\right)\right]\right\}$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$

$$a_n = \frac{2}{\pi n^2}\left[2\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n\right]$$

when n is odd,

$a_n = 0$ i.e., $a_1 = a_3 = a_5 = \dots = 0$

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when n is even,

$$a_2 = \frac{2}{2^2\pi}\left[2\cos\left(\frac{2\pi}{2}\right)-1-1\right] = \frac{2}{2^2\pi}\left[-2-2\right]$$

$$= \frac{-8}{2^2\pi} = \frac{-2}{\pi.1^2} \qquad \boxed{a_2 = \frac{-2}{\pi.1^2}}$$

$$a_4 = \frac{2}{4^2\pi}\left[2\cos\left(\frac{4\pi}{2}\right)-1-1\right] = \frac{2}{4^2\pi}\left[2-2\right]$$

$$\boxed{a_4 = 0}$$

$$a_6 = \frac{2}{6^2\pi}\left[2\cos\left(\frac{6\pi}{2}\right)-1-1\right] = \frac{2}{6^2\pi}\left[-2-2\right]$$

$$= \frac{-8}{6^2\pi} = \frac{-2}{\pi.3^2} \qquad \boxed{a_6 = \frac{-2}{\pi.3^2}}$$

Step 4: Cosine Series of $f(x)$ in $(0,\pi)$

$$f(x) = \frac{\pi/2}{2} + \cancel{\frac{2}{\pi}}\left[\sum_{n=2,4,6}^{\infty} a_n \cos nx\right]$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \cdots\right]$$